ETH zürich

Swiss Federal Institute of Technology Zürich
Department of Mathematics
Bachelor Thesis

An Introduction to the Dynamics of Compact Group Automorphisms after Klaus Schmidt

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$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[loop above, "\alpha_{\mathbf{n}}^{Y}"] \arrow[r, hook, "\iota"] \arrow[d, leftrightarrow, "\widehat{\cdot}" description] & X \arrow[loop above, "\alpha_{\mathbf{n}}"] \arrow[r, two heads, "\pi"] \arrow[d, leftrightarrow, "\widehat{\cdot}" description] & X/Y \arrow[loop above, "\alpha_{\mathbf{n}}^{X/Y}"] \arrow[r] \arrow[d, leftrightarrow, "\widehat{\cdot}" description] & 0 \\
0 & M/Y^{\perp} \arrow[l] \arrow[loop below, "\cdot u^{\mathbf{n}}"'] & M \arrow[l, two heads, "\hat{\iota}"'] \arrow[loop below, "\cdot u^{\mathbf{n}}"'] & Y^{\perp} \arrow[l, hook', "\hat{\pi}"'] \arrow[loop below, "\cdot u^{\mathbf{n}}"'] & 0 \arrow[l]
\end{tikzcd}$$

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Dedicated to the memory of my dear friend, Gian.

Abstract

This thesis provides an introduction to the theory of algebraic dynamical systems based on the work of Klaus Schmidt. We begin by establishing fundamental dynamical properties through the lens of actions of countable groups on compact metrizable groups by continuous automorphisms, which we call *quasi-algebraic actions*. Specifically, we introduce the notions of topological transitivity, ergodicity, mixing, and expansiveness, and illustrate them through four running examples. With these foundations in place, we explore the rigidity of quasi-algebraic actions, structural constraints imposed by dynamical conditions, and provide a spectral characterization of ergodicity and mixing. The culmination of this work is an "algebraic dictionary" that establishes a correspondence between algebraic $\mathbb{Z}^d$-actions and countable modules over a Laurent polynomial ring, translating dynamical properties into algebraic ones and providing a unified framework for their study.

Acknowledgements

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Contents

Introduction

What is a Dynamical System?

In its most rudimentary form, a dynamical system is a set X together with a self-map $T : X \to X$. Already, several natural questions arise. For a given point $x \in X$, how does it "evolve" under T, i.e., how does the sequence $(x, T(x), T^2(x), \dots)$ behave? Does the sequence become stationary? Does it ever repeat? Does it "cover" X? Do points with such properties even exist?

However, to allow for systematic study, we usually impose additional structure on our set and the accompanying map. Various constraints have been studied extensively. For example, one may:

- Take X to be a compact metric space and T a homeomorphism of X (*Topological Dynamics*).
- Take X to be a smooth manifold and T a diffeomorphism of X (*Smooth Dynamics*).
- Take X to be a probability space and T a measure-preserving transformation (*Measurable Dynamics* or *Ergodic Theory*).
- Take X to be the quotient $\Gamma\backslash G$ of a Lie group G by a lattice Γ, and T a translation by an element of G (*Homogeneous Dynamics*).
- Take X to be the Riemann sphere $\mathbb{C}P^1$ and T a rational function on X (*Complex Dynamics*).

The boundaries between these classes are not strict. Indeed, one often studies systems that naturally fit into several of the above frameworks.

The repeated application of T is often interpreted as the passage of time in discrete steps within the space X. With this in mind, it seems natural to ask whether we may extend the notion of a dynamical system to include the case of spaces that evolve continuously over time. Indeed, in a more abstract sense, a dynamical system given by a single transformation $T : X \to X$ may also be viewed as an action of the monoid $\mathbb{Z}_{\geq 0}$ on X given by $k \mapsto T^k$ for $k \in \mathbb{Z}_{\geq 0}$, or, if T is bijective, as an action of the group $\mathbb{Z}$. An action of $\mathbb{R}$ on a space X then provides the desired extension to continuous time. Leaving the interpretation of time behind and allowing for all kinds of symmetries, one can (and we will) consider an action of an arbitrary group Γ (or monoid) on a space X.

In the more general setting of a group Γ acting on a space X, one usually imposes additional structure on Γ, X, and the action, just as in the case of a single transformation. One class that will be of particular interest to us is that of actions of countable groups on compact metrizable groups by continuous automorphisms. Before proceeding further, let us consider several motivating examples.

Example 0.1 (Arnold's Cat Map)**.** We consider the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the underlying space and a single transformation $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ induced by left multiplication by the invertible integer matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

i.e., $T_A(x) = Ax$. This dynamical system exhibits a variety of chaotic behaviors, which we will rigorously formalize in subsequent chapters. For now, we build intuition through a few visual experiments.

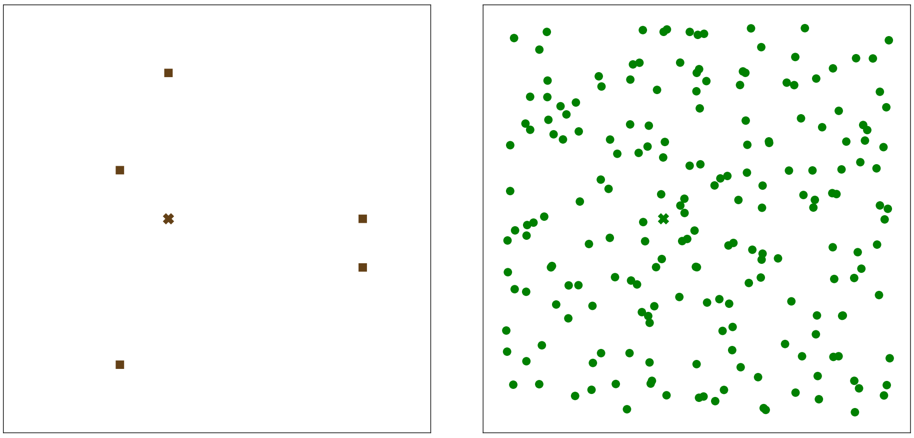


Figure 1: The plot shows 100 iterations of the point $p_1 = (3/8, 1/2) + \mathbb{Z}^2$ on the left and the point $p_2 = (\sqrt{2}, 1/2) + \mathbb{Z}^2$ on the right under Arnold's Cat Map, that is, $T_A^k(p_i)$ for $k = 0, \ldots, 99$ (the starting points p_1 and p_2 are marked by crosses).

Recall that we may think of the 2-torus as the unit square with certain edge identifications. In Fig. 1, we see the evolution of two points under the application of T_A. Although they start out relatively close to each other, they seem to behave very differently. The brown point p_1 on the left only ever visits six distinct points, while the green point p_2 on the right seems to have spread across the entire 2-torus. We will later develop language to describe behaviors such as these.

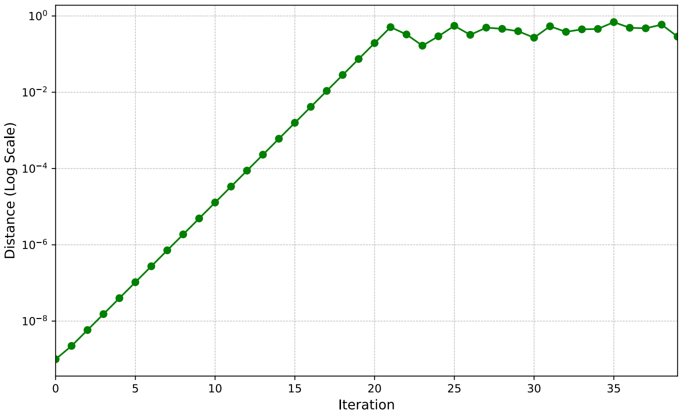


Figure 2: The plot shows how the distance between the points $(\pi, \sqrt{2}) + \mathbb{Z}^2$ and $(\pi + 10^{-9}, \sqrt{2}) + \mathbb{Z}^2$ evolves under Arnold's Cat Map.

In Fig. 2, we observe that two points that are initially very close to each other will diverge under T_A at an exponential rate until their distance is “saturated”.

Finally, Fig. 3 illustrates how T_A "stirs up" the space.

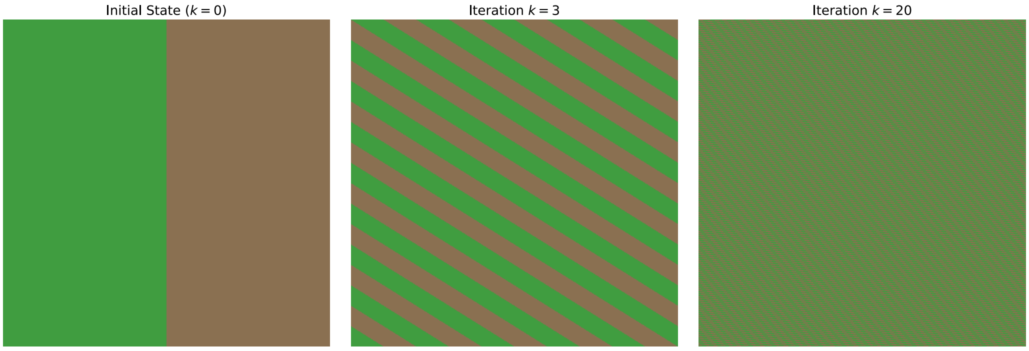


Figure 3: This figure shows how a coloring of the 2-torus mixes under the application of Arnold's Cat Map.

Example 0.2 (Toral Rotation)**.** We again consider the 2-torus $\mathbb{T}^2$ as the underlying space and a single transformation $T_R : \mathbb{T}^2 \to \mathbb{T}^2$ induced by left multiplication by

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thinking of R as a transformation of $\mathbb{R}^2$, we see that it is a rotation by $\pi/2$. In particular, we have that R^4 is equal to the identity matrix. Thus, every point $x \in \mathbb{T}^2$ will repeatedly return to its starting position under T_R, that is, $x = T_R^4(x) = T_R^{4k}(x)$ for all integers $k \geq 1$. This is an example of a very well-behaved map, contrasting Example 0.1. Indeed, the experiments done in the previous example would not yield any interesting results in this case.

Example 0.3 (Times Two Map)**.** For this example, we consider the 1-torus $\mathbb{T}$ and a single transformation $T_2 : \mathbb{T} \to \mathbb{T}$ defined by $x \mapsto 2x$. This dynamical system is tightly linked to the binary expansion. Indeed, if $x = \sum_{i=1}^{\infty} a_i 2^{-i} + \mathbb{Z}$ for some $a_i \in \{0, 1\}$, then

$$\begin{aligned} T_2^k(x) &= 2^k x \\ &= \underbrace{\sum_{i=1}^{k} a_i 2^{-i+k}}_{\in \mathbb{Z}} + \sum_{i=k+1}^{\infty} a_i 2^{-i+k} + \mathbb{Z} \\ &= \sum_{i=1}^{\infty} a_{i+k} 2^{-i} + \mathbb{Z}, \end{aligned}$$

for any integer $k \geq 0$. Thus, k applications of the transformation T_2 correspond to moving the radix point in the binary expansion by k digits to the right and discarding the integer part. This observation enables us to explicitly study the behavior of different points under T_2.

Consider any point $x \in \mathbb{T}$ with a finite binary expansion, i.e., $x = \sum_{i=1}^{N} a_i 2^{-i} + \mathbb{Z}$ with $a_i \in \{0, 1\}$ and $N \in \mathbb{Z}_{\geq 1}$. Then, using the above, we see that $T_2^N(x) = 0$. Thus, the system $T_2 : \mathbb{T} \to \mathbb{T}$ has infinitely many points that go to zero in finite time.

Another interesting class of points consists of those with a periodic binary expansion, i.e., points $x \in \mathbb{T}$ such that $x = \sum_{i=0}^{\infty} a_{i \bmod N} 2^{-i-1} + \mathbb{Z}$ for some $a_0, \ldots, a_{N-1} \in \{0, 1\}$. These points will repeatedly visit the same finite number of points under T_2. For example, for x as above, we have $T_2^k(x) = T_2^{k \bmod N}(x)$ for all integers $k \geq 0$. To make it even more explicit, consider $x = 1/5 + \mathbb{Z}$. Its binary expansion is given by $0.\overline{0011}$ (i.e., $0.001100110011\ldots$) and we thus expect it to return to its starting point every four iterations of T_2. Indeed, calculating the trajectory of $x = 1/5 + \mathbb{Z}$ under T_2, we find

$$\begin{aligned} T_2(x) &= 2/5 + \mathbb{Z} \\ T_2^2(x) &= 4/5 + \mathbb{Z} \\ T_2^3(x) &= 3/5 + \mathbb{Z} \\ T_2^4(x) &= 1/5 + \mathbb{Z}, \end{aligned}$$

as expected. This shows that there are infinitely many points that return to their starting position after a finite number of iterations. In fact, one may check that this set of points is dense in $\mathbb{T}$.

Example 0.4 (Square Shift)**.** We consider the space

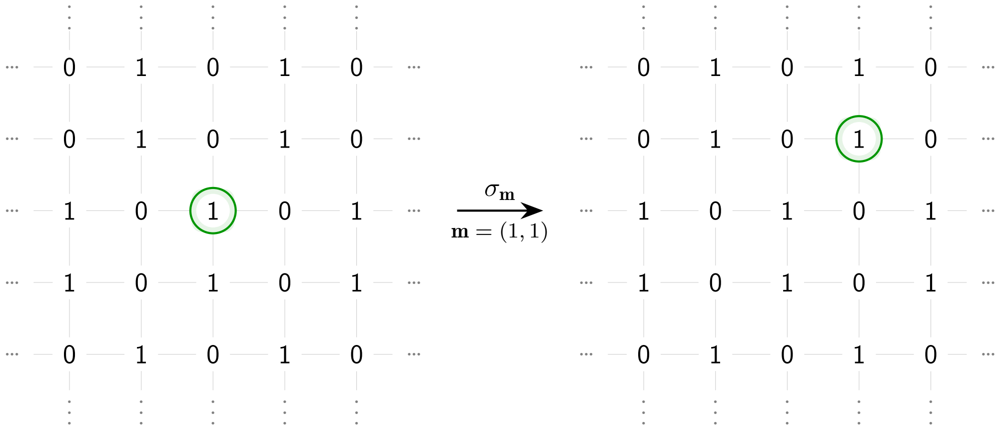
$$X = \left\{ (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}+(1,1)} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2 \right\}.$$

There is a natural action σ of $\mathbb{Z}^2$ on X given by "shifting" the coordinates, usually referred to as the shift-action,

$$\sigma_{\mathbf{m}}((x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2}) = (x_{\mathbf{n}+\mathbf{m}})_{\mathbf{n} \in \mathbb{Z}^2},$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $(x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in X$.

We may think of each point in the space X as an infinite two-dimensional grid, where each vertex has a value in $\mathbb{Z}/2\mathbb{Z}$ constrained by the local rule specified in the definition of X. From this perspective, the action σ corresponds to moving the "origin vertex" of the grid.



Original configuration x **Shifted configuration** $\sigma_{\mathbf{m}}(x)$

Figure 4: The Square Shift (X, σ) viewed as a grid of values in $\mathbb{Z}/2\mathbb{Z}$, where the action corresponds to moving the origin vertex, which is marked with a green circle.

The four examples above will accompany us throughout this thesis. They are meant to serve as a source of intuition and motivation for the theory that we will develop.

The Goal of this Thesis

In the preceding section, we introduced four distinct dynamical systems: Arnold's Cat Map, the Toral Rotation, the Times Two Map, and the Square Shift. As we will see in Chapter 1, analyzing the behavior of dynamical systems often requires a variety of ad hoc methods. The primary goal of this thesis is to present an algebraic framework that allows us to analyze the behavior of algebraic $\mathbb{Z}^d$-actions in a unified and systematic way.

This theory was primarily developed by Klaus Schmidt, and our exposition is based on his monograph *Dynamical Systems of Algebraic Origin* [Sch95].

Prerequisites

We assume the reader is familiar with the standard undergraduate curriculum in mathematics, specifically real analysis, abstract algebra (including linear algebra and the basics of groups, rings, fields, and modules), point-set topology, measure theory, and the basics of Fourier analysis. To keep this thesis reasonably self-contained, we have collected the more specialized background material required for our proofs in the appendices. This material includes the fundamental theory of topological groups and Pontryagin duality (Appendix A), unitary representations leading up to the Peter–Weyl theorem (Appendix B), topics in commutative algebra, such as primary decomposition and associated prime ideals (Appendix C), and a brief overview of the Gelfand transform for Banach algebras (Appendix D). We reference the results from these appendices whenever they are used.

Outline

The remainder of this thesis is structured as follows.

Chapter 1 We start by introducing the class of dynamical systems that we will study, namely, quasi-algebraic actions. Through the lens of quasi-algebraic actions, we then define and characterize the fundamental dynamical properties that are of interest to us: topological transitivity, ergodicity, mixing, expansiveness, and the descending chain condition. To illustrate these concepts and build intuition, we will regularly return to our four running examples introduced above. The content of this chapter is standard within the fields of dynamical systems and ergodic theory [EW11; ES14; KH95; Kit98].

Chapter 2 Having established the basic vocabulary, we turn our attention to the structural rigidity of quasi-algebraic actions. We will demonstrate how dynamical properties, such as expansiveness and the descending chain condition, force a system to be conjugate to a highly structured Lie subshift. Furthermore, we will leverage the Peter–Weyl theorem to provide spectral characterizations of ergodicity and mixing. The theory presented in this chapter was established by Bruce Kitchens and Klaus Schmidt [KS89].

Chapter 3 Building on the results from the preceding chapter, we turn to the main goal of this thesis. We will establish a correspondence between algebraic $\mathbb{Z}^d$-actions and countable modules over the ring of Laurent polynomials with integer coefficients in d commuting variables. Using this correspondence, we will then build a dictionary between the dynamical properties of such a system and the algebraic properties of the corresponding module. Finally, we will apply this unified framework to re-examine our four running examples and explore several additional systems that further highlight the strengths of this theory. The results presented in this chapter were developed by Klaus Schmidt [Sch90] based on his collaborative work with Bruce Kitchens [KS89].

Outlook Finally, we will provide pointers for the interested reader to continue the exploration by offering a glimpse into the later chapters of Klaus Schmidt's monograph [Sch95]. We will briefly discuss how the algebraic dictionary we developed may be extended to include other dynamical properties, in particular, higher-order mixing and entropy, and illustrate how it effectively bridges distinct areas of mathematics.

Chapter 1

The Language of Dynamics & Quasi-Algebraic Actions

The goal of this chapter is to establish the fundamental vocabulary for describing dynamical systems through the lens of quasi-algebraic actions. We will define and characterize the dynamical properties that are of interest to us and regularly return to the four examples introduced in the Introduction to illustrate these concepts.

Before we begin, we note that the content of this chapter is standard within ergodic theory and dynamical systems, and we refer the reader to [EW11; ES14; KH95; Kit98].

Definition 1.1. A *quasi-algebraic action* is an action α of a countably infinite group Γ[1] on a compact metrizable group X by continuous automorphisms of X. Concretely, it is a group homomorphism $\alpha : \Gamma \to \operatorname{Aut}(X)$[2] and we write α_γ instead of $\alpha(\gamma)$ for $\gamma \in \Gamma$.

Remark 1.2. We note that the term "quasi-algebraic action" is non-standard and is introduced here specifically for the purposes of this thesis as a derivative of the well-established term "algebraic action" (see Definition 3.1). It provides a concise way to refer to the systems that are of interest to us. The descriptor "quasi" indicates that these systems are not fully "algebraic": Since we do not require the compact metrizable group X to be abelian, its unitary dual $\widehat{X}$ does not admit a natural group structure.

Definition 1.3. Let $\alpha : \Gamma \to \operatorname{Aut}(X)$ be a quasi-algebraic action. A subset $A \subseteq X$ is called *α-invariant* if $\alpha_\gamma^{-1}(A) \subseteq A$ for every $\gamma \in \Gamma$.

Let $\alpha : \Gamma \to \operatorname{Aut}(X)$ be a quasi-algebraic action. If $Y \subseteq X$ is a closed, α-invariant subgroup, then by restricting α to Y, we obtain a quasi-algebraic action of Γ on Y, which we will denote by α^Y. If Y is additionally normal in X, then there is also an induced quasi-algebraic action on the quotient X/Y, denoted by $\alpha^{X/Y}$.

Definition 1.4. Let $\alpha : \Gamma \to \operatorname{Aut}(X)$ and $\beta : \Gamma \to \operatorname{Aut}(Y)$ be two quasi-algebraic actions of Γ. Then we call α and β *conjugate* if there exists a continuous group isomorphism $\phi : X \to Y$ such that $\phi \circ \alpha_\gamma = \beta_\gamma \circ \phi$ for every $\gamma \in \Gamma$.

If two quasi-algebraic actions are conjugate to each other, then, for all practical purposes, they are the same dynamical system. In particular, they share all dynamical properties that we will define in the following sections.

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[1] The reader is invited to consider what happens if Γ is a finite group throughout this chapter and thereby convince themselves that finite actions are not of particular interest to us.

[2] For an arbitrary topological group G, we use the notation $\operatorname{Aut}(G)$ to mean the set of all bi-continuous group automorphisms of G, i.e., the set of all continuous bijective group homomorphisms $\phi : G \to G$ such that ϕ^{-1} is also continuous.

1.1 Topological Transitivity

For the following definitions, we consider a quasi-algebraic action $\alpha : \Gamma \to \mathrm{Aut}(X)$.

Definition 1.5. We define the *α-orbit* (or simply *orbit* if no ambiguity arises) of a point $x \in X$ as the set $\alpha_\Gamma(x) := \{\alpha_\gamma(x) : \gamma \in \Gamma\}$.

Definition 1.6. We call a point $x \in X$ *α-periodic* (or simply *periodic*) if its orbit $\alpha_\Gamma(x)$ is a finite set.

If $\Gamma = \mathbb{Z}$, a point $x \in X$ having a finite orbit is equivalent to the existence of an integer $k \geq 1$ such that $\alpha_k(x) = x$, which aligns with the classical intuition of periodicity. Furthermore, we observe that since $\alpha_\Gamma(xy^{-1}) \subseteq \alpha_\Gamma(x)\alpha_\Gamma(y)^{-1}$ for any $x, y \in X$, the set of periodic points of α forms a subgroup of X.

Definition 1.7. The action α is said to be *topologically transitive* if there exists a point $x \in X$ whose orbit is dense in X.

Topological transitivity is perhaps best thought of as a topological indecomposability condition: The space X cannot be decomposed into disjoint, non-empty, open and invariant subsets. Definition 1.7 formulates this via the existence of a single point that eventually visits the entire space. The following proposition provides equivalent characterizations that are often easier to work with.

Proposition 1.8. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Then the following are equivalent:*

(a) The action α is topologically transitive.
(b) For any non-empty α-invariant open subset $U \subseteq X$, its closure $\overline{U}$ is equal to X.
(c) For any two non-empty open subsets $U, V \subseteq X$, there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma^{-1}(U) \cap V \neq \varnothing$.
(d) The set of points with a dense orbit $\{x \in X : \overline{\alpha_\Gamma(x)} = X\}$ is a dense G_δ-set in X.[3]

Proof. (a) $\Longrightarrow$ (b): Let $U \subseteq X$ be a non-empty, α-invariant, and open subset. By our assumption, there exists a point $x_0 \in X$ with a dense orbit. Since U is non-empty and open, we have $\alpha_\Gamma(x_0) \cap U \neq \varnothing$, i.e., there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma(x_0) \in U$. By α-invariance of U, we obtain that $\alpha_\Gamma(x_0) \subseteq U$, and finally, density of $\alpha_\Gamma(x_0)$ implies $\overline{U} = X$.

(b) $\Longrightarrow$ (c): Let $U, V \subseteq X$ be non-empty, open subsets. Set $\tilde{U} = \bigcup_{\gamma \in \Gamma} \alpha_\gamma^{-1}(U)$. Then $\tilde{U}$ is a non-empty, open, α-invariant subset and hence dense in X, by our assumption. We therefore have $\tilde{U} \cap V \neq \varnothing$, so there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma^{-1}(U) \cap V \neq \varnothing$, as desired.

(c) $\Longrightarrow$ (d): Since X is compact and metrizable, there is a countable base $(B_n)_{n \in \mathbb{N}}$ of non-empty open sets for the topology of X. By our assumption, the sets

$$\tilde{B}_n = \bigcup_{\gamma \in \Gamma} \alpha_\gamma^{-1}(B_n)$$

are dense in X. Now, the Baire category theorem implies that $G = \bigcap_{n \in \mathbb{N}} \tilde{B}_n$ is a dense G_δ-set in X. We claim that any point in G has a dense orbit, which would conclude the proof of this implication. Indeed, let $x \in G$ and $U \subseteq X$ be a non-empty open subset. Since $(B_n)_{n \in \mathbb{N}}$ forms a base, there exists an $m \in \mathbb{N}$ such that $B_m \subseteq U$. As x lies in G, there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma(x) \in B_m$, and hence $\alpha_\Gamma(x) \cap U \neq \varnothing$.

(d) $\Longrightarrow$ (a): Any dense subset of X is non-empty. □

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[3] In a topological space, a G_δ-set is a subset of the space that is a countable intersection of open sets.

Example 1.9 (Arnold's Cat Map II)**.** We revisit the dynamical system $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by matrix multiplication with $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ from Example 0.1.

Observe that with the topology and additive group structure inherited from $\mathbb{R}^2$, $\mathbb{T}^2$ is a compact metrizable (and abelian) group and the map T_A is a continuous group automorphism. Thus, T_A is an example of a quasi-algebraic action.[4]

We claim that the set of periodic points of T_A is precisely given by $\mathbb{Q}^2/\mathbb{Z}^2$, and in particular, it is dense in $\mathbb{T}^2$. Indeed, assume that $x \in \mathbb{T}^2$ is periodic, i.e., there exists an integer $n \geq 1$ such that $T_A^n(x) = x$. Let $\tilde{x}$ be any representative of x in $\mathbb{R}^2$. Then $(A^n - I)\tilde{x} = k$ for some integer vector $k \in \mathbb{Z}^2$, where I denotes the identity matrix. The eigenvalues of A are $\frac{3 \pm \sqrt{5}}{2}$, which implies that $A^n - I$ is invertible over $\mathbb{Q}$. Hence, $\tilde{x} = (A^n - I)^{-1}k$ lies in $\mathbb{Q}^2$ and we obtain $x \in \mathbb{Q}^2/\mathbb{Z}^2$. For the reverse inclusion, let $d \in \mathbb{Z}_{\geq 1}$ and consider the set

$$O_d = \{x \in \mathbb{T}^2 : d \cdot x = 0\}.$$

It is readily checked that this set is finite (of size d^2) and that $T_A(O_d) = O_d$. Now, since for any $x \in \mathbb{Q}^2/\mathbb{Z}^2$ we may find a $d \in \mathbb{Z}_{\geq 1}$ such that $x \in O_d$, we see that x is periodic under T_A, proving our claim.

Next, we demonstrate that T_A is topologically transitive.[5] We use part (c) of Proposition 1.8 to prove this claim. Let $W^u, W^s \subseteq \mathbb{T}^2$ be the images of the eigenspaces $E^u, E^s \subseteq \mathbb{R}^2$ under the canonical projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ corresponding to the eigenvalues $\lambda_u = \frac{3+\sqrt{5}}{2}$, $\lambda_s = \frac{3-\sqrt{5}}{2}$ respectively. For $x \in \mathbb{T}^2$, we define the stable manifold at x as

$$M^s(x) = \{y \in \mathbb{T}^2 : d_{\mathbb{T}^2}(T_A^n(x), T_A^n(y)) \to 0 \text{ as } n \to \infty\}$$

and the unstable manifold at x as

$$M^u(x) = \{y \in \mathbb{T}^2 : d_{\mathbb{T}^2}(T_A^{-n}(x), T_A^{-n}(y)) \to 0 \text{ as } n \to \infty\}$$

where $d_{\mathbb{T}^2}(y, z) = \min_{k \in \mathbb{Z}^2} \|\tilde{y} - \tilde{z} + k\|_2$ for $y, z \in \mathbb{T}^2$ with representatives $\tilde{y}$, $\tilde{z}$ in $\mathbb{R}^2$.

We wish to show that $W^* + x = M^*(x)$ for all $x \in \mathbb{T}^2$ and $* \in \{u, s\}$. First, notice that since T_A is a group homomorphism, we have $M^*(x) = M^*(0) + x$, so it suffices to prove the equality for $x = 0$. We first consider the stable manifold. Let $v \in E^s$. Then for any $n \geq 1$, we have

$$d_{\mathbb{T}^2}(0, T_A^n(\pi(v))) = d_{\mathbb{T}^2}(0, \pi(A^n v)) \leq \|A^n v\| = |\lambda_s|^n \|v\|.$$

Since $|\lambda_s| < 1$, the right-hand side converges to 0 as $n \to \infty$, which implies $\pi(E^s) \subseteq M^s(0)$. For the reverse inclusion, let $x \in M^s(0)$. We choose $\delta > 0$ sufficiently small such that the projection π restricted to the ball $B_\delta(0)$ is an invertible isometry and, crucially, that $\|A\|\delta < 1/2$, where $\|A\|$ denotes the operator norm of A. The second condition ensures that a vector $v \in B_\delta(0)$ cannot be mapped to $B_\delta(k)$ for any non-zero $k \in \mathbb{Z}^2$, because $\|Av\| \leq \|A\|\|v\| < \|A\|\delta < 1/2$. Since $x \in M^s(0)$, there exists an integer $N \geq 1$ such that $y = T_A^k(x)$ lies in $\pi(B_\delta(0))$ for all $k \geq N$. Let $\tilde{y}$ be the unique representative of y in $\mathbb{R}^2$ such that $\tilde{y} \in B_\delta(0)$. Then, combining the convergence of $d_{\mathbb{T}^2}(0, T_A^n(y))$ to 0 as $n \to \infty$ with our choice of δ, we obtain that $A^n\tilde{y} \in B_\delta(0)$ for all $n \geq 0$. Since $|\lambda_u| > 1$, we now see

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[4] We admit that calling T_A a quasi-algebraic action is a slight abuse of terminology. The toral automorphism T_A merely induces a quasi-algebraic action given by $\mathbb{Z} \ni k \mapsto T_A^k \in \mathrm{Aut}(\mathbb{T}^2)$. Nevertheless, we will continue to commit this slight abuse.

[5] The geometric argument presented here, relying on stable and unstable manifolds, is known as Hopf's argument. It was first introduced by Eberhard Hopf to prove ergodicity of the geodesic flow [Hop39].

by decomposing $\tilde{y}$ with respect to an eigenbasis of A that $\tilde{y}$ must lie in E^s, as otherwise $\|A^n\tilde{y}\|$ would go to infinity as $n \to \infty$. This shows $M^s(0) \subseteq \pi(E^s)$ and we thus obtain equality. The case of the unstable manifold follows an analogous argument by considering the backward iterations T_A^{-n}.

Now let $U, V \subseteq \mathbb{T}^2$ be non-empty open subsets. We have seen above that the periodic points are dense in $\mathbb{T}^2$, so we may find periodic points $p \in U$ and $q \in V$. Let m be a common period of p and q, i.e., a positive integer such that $T_A^m(p) = p$ and $T_A^m(q) = q$. Then both $W^u + p$ and $W^s + q$ are invariant under T_A^m. Furthermore, by lifting to $\mathbb{R}^2$, we see that they have a non-empty intersection (their lifts are two non-parallel lines and thus intersect). Let r be a point in the intersection $(W^u + p) \cap (W^s + q)$. Then, by our observation above, we have $d_{\mathbb{T}^2}(T_A^{-mn}(r), p) \to 0$ and $d_{\mathbb{T}^2}(T_A^{mn}(r), q) \to 0$ as $n \to \infty$. Thus, for large n, we have $T_A^{-mn}(r) \in U$ and $T_A^{mn}(r) \in V$, so that $T_A^{2mn}(U) \cap V$ is non-empty.

Remark 1.10. Notice that to prove both claims in Example 1.9, we only relied on the fact that A did not have any eigenvalues of modulus 1. Indeed, the following holds (and may be proven analogously). Let $T_B : \mathbb{T}^d \to \mathbb{T}^d$ be a toral automorphism given by matrix multiplication with $B \in \mathrm{GL}_d(\mathbb{Z})$. If B has no eigenvalues of modulus 1, then T_B is topologically transitive.

Example 1.11 (Toral Rotation II)**.** We revisit Example 0.2. By the same reasoning as in Example 1.9, we see that T_R is a quasi-algebraic action. We had already noted that $T_R^4(x) = x$ for any $x \in \mathbb{T}^2$. This shows that every point of $\mathbb{T}^2$ is periodic with an orbit of size dividing 4 under T_R. Furthermore, this implies that T_R is not topologically transitive.

Remark 1.12. Before we revisit the Times Two Map, we must address the fact that it is not a quasi-algebraic action. For the context of this remark, we define a *monoid action* to be an action of a countable monoid on a compact metrizable group by continuous surjective endomorphisms. If we denote by $T_2 : \mathbb{T} \to \mathbb{T}$ the Times Two Map and define $\beta : \mathbb{Z}_{\geq 0} \to \mathrm{End}(\mathbb{T})$ by $k \mapsto T_2^k$, we see that the Times Two Map is a monoid action. However, since T_2 is not injective, the Times Two Map is not a quasi-algebraic action.

We have intentionally formulated the exposition of Chapter 1 so that every definition and result applies verbatim to monoid actions, avoiding any reliance on the invertibility of the maps. For instance, the reader may confirm that the definition of topological transitivity (Definition 1.7) and its equivalent characterizations (Proposition 1.8) make sense and hold in this broader context. Thus, Chapter 1 could have been developed for the strictly larger class of monoid actions. Nevertheless, for pedagogical reasons and since Chapter 2 strictly relies on invertibility, we chose to develop the material through the lens of quasi-algebraic actions.

Furthermore, we decided to include the Times Two Map, despite it not being a quasi-algebraic action, for two primary reasons. First, the Times Two Map is a classical example in dynamical systems and ergodic theory, making it natural to include in our discussion. Second, it is closely related to a quasi-algebraic action via the natural extension of a monoid action, providing an opportunity to discuss this construction (see Example 3.24).

Example 1.13 (Times Two Map II)**.** We return to the Times Two Map $T_2 : \mathbb{T} \to \mathbb{T}$ given by $x \mapsto 2x$ from Example 0.3. Our goal is to show that T_2 is topologically transitive. In this case, we can do "better" than in Example 1.9 and explicitly construct a point x_0 in $\mathbb{T}$ with a dense orbit, i.e., for any $x \in \mathbb{T}$ and any $\varepsilon > 0$ there exists an integer $k \geq 0$ such that $d_{\mathbb{T}}(T_2^k(x_0), x) < \varepsilon$, where the distance is defined by $d_{\mathbb{T}}(x, y) := \min_{n \in \mathbb{Z}} |x - y + n|$ for $x, y \in \mathbb{T}$.

Let $(w_n)_{n \geq 1}$ be an enumeration of all finite words in the alphabet $\{0, 1\}$ and define

$$x_0 = 0.w_1 w_2 w_3 \cdots + \mathbb{Z}$$

in binary expansion. Now, given $x = \sum_{i=1}^{\infty} a_i 2^{-i}$ in $\mathbb{T}$, for some $a_i \in \{0,1\}$, and $\varepsilon > 0$, choose N large such that $2^{-N} < \varepsilon$ and consider the finite word $w = a_1 a_2 \dots a_N$. Recall that application of T_2 corresponds to shifting the binary expansion. So by construction of x_0, w appears as one of the words w_n and thus there exists an integer $k \geq 0$ such that the binary expansion of $T_2^k(x_0)$ also starts with w. This implies $d_{\mathbb{T}}(T_2^k(x_0), x) \leq 2^{-N} < \varepsilon$, proving our claim.

Example 1.14 (Square Shift II)**.** Finally, we also revisit Example 0.4, where we introduced the shift-action σ on the space

$$X = \{(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}+(1,1)} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

We note that there is a natural group structure on X given by component-wise addition, and as such, it is a subgroup of $Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$. If we equip $\mathbb{Z}/2\mathbb{Z}$ with the discrete topology, Tychonoff's theorem implies that Y is a compact group. Since Y is clearly second-countable, it is also metrizable by the Birkhoff–Kakutani characterization. To be more concrete, one may check that if we enumerate $\mathbb{Z}^2$ as $(\mathbf{n}_k)_{k\geq 1}$ and define

$$d((x_{\mathbf{n}}), (y_{\mathbf{n}})) = \sum_{k\geq 1} \frac{|x_{\mathbf{n}_k} - y_{\mathbf{n}_k}|}{2^k},$$

then d is a metric on Y, which induces the product topology. Since X is a closed subset of Y, we obtain that X is a compact metrizable group as well. We note that the so-called *cylinder sets* of the form

$$[c]_F := \{(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2} \in X : x_{\mathbf{n}} = c_{\mathbf{n}} \text{ for all } \mathbf{n} \in F\},$$

where F is a *finite* subset of $\mathbb{Z}^2$ and $c \in (\mathbb{Z}/2\mathbb{Z})^F$, form a base for the product topology.[6] Using this base, it is readily checked that every $\sigma_{\mathbf{m}}$ is a continuous group automorphism of X, for each $\mathbf{m} \in \mathbb{Z}^2$. Thus σ is a quasi-algebraic action.

We claim that σ is topologically transitive. We will use part (c) of Proposition 1.8 to prove our claim. Let U, V be non-empty open subsets of X. By the above, there exist finite subsets F_U, F_V of $\mathbb{Z}^2$ and elements $c^U \in (\mathbb{Z}/2\mathbb{Z})^{F_U}$, $c^V \in (\mathbb{Z}/2\mathbb{Z})^{F_V}$ such that $\varnothing \neq [c^U]_{F_U} \subseteq U$ and $\varnothing \neq [c^V]_{F_V} \subseteq V$. Since $[c^U]_{F_U} \neq \varnothing$ and $[c^V]_{F_V} \neq \varnothing$, the finite "patterns" c^U and c^V are admissible in the sense that they satisfy the local rule specified in the definition of X and can be extended to points in X. We now note that given any two finite admissible patterns placed, for example, diagonally to each other, we can combine them to form a new admissible pattern (see Fig. 1.1). Thus, we may find an $\mathbf{m} \in \mathbb{Z}^2$ such that F_U and $F_V + \mathbf{m}$ are diagonal to each other.[7] Thus, we have that

$$\begin{aligned} &\varnothing \neq [c^U]_{F_U} \cap [(c^V_{\mathbf{n}-\mathbf{m}})_{\mathbf{n}\in F_V+\mathbf{m}}]_{F_V+\mathbf{m}} \\ &\quad = \{(x_{\mathbf{n}}) \in X : x_{\mathbf{n}} = c^U_{\mathbf{n}} \text{ and } x_{\tilde{\mathbf{n}}} = c^V_{\tilde{\mathbf{n}}-\mathbf{m}} \text{ for all } \mathbf{n} \in F_U \text{ and } \tilde{\mathbf{n}} \in F_V + \mathbf{m}\}. \end{aligned}$$

Let x be an element in the above intersection. Then we have that $x \in [c^U]_{F_U} \subseteq U$ and $\sigma_{\mathbf{m}}(x) \in [c^V]_{F_V} \subseteq V$, which shows that $U \cap \sigma_{-\mathbf{m}}(V) \neq \varnothing$, proving our claim.

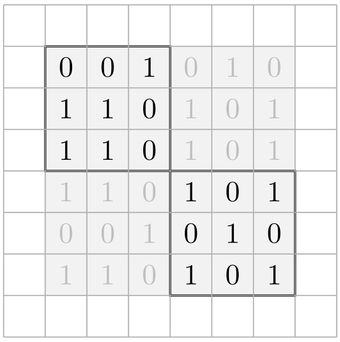
Next, we wish to show that the set of σ-periodic points is dense in X. We first claim that any point $x \in X$ is fully determined by its values on the "axes", i.e., by $(x_{\mathbf{n}})_{\mathbf{n}\in\mathcal{T}}$, where $\mathcal{T} = (\{0\} \times \mathbb{Z}) \cup (\mathbb{Z} \times \{0\})$. Specifically, we will prove that

$$x_{i,j} = x_{0,0} + x_{i,0} + x_{0,j}, \tag{1.1}$$

---

[6] This observation is nothing more than a translation of the definition of the product topology to our specific case.

[7] Note that F_U and F_V might not be rectangles in $\mathbb{Z}^2$, but we may always find a finite rectangle enclosing them.



0	0	1	0	1	0
1	1	0	1	0	1
1	1	0	1	0	1
1	1	0	1	0	1
0	0	1	0	1	0
1	1	0	1	0	1

Figure 1.1: Gluing two finite admissible patterns in X.

for any $i, j \in \mathbb{Z}$. First, notice that it trivially holds in the case where i or j is equal to zero. Next, notice that, by symmetry, it is enough to prove it for $i, j \geq 1$. We can do this by a staggered induction on j and i. Notice that the local rule of X implies $x_{1,1} = x_{0,0}+x_{1,0}+x_{0,1}$. This shows the case $i = j = 1$. Let $i > 1$ and assume Eq. (1.1) holds for $i-1$ and $j = 1$. Then, again by the local rule of X, we have $x_{i,1} = x_{i-1,1} + x_{i,0} + x_{i-1,0}$. Plugging in our assumption, we obtain

$$\begin{aligned} x_{i,1} &= x_{0,0} + x_{i-1,0} + x_{0,1} + x_{i,0} + x_{i-1,0} \\ &= x_{0,0} + x_{i,0} + x_{0,1}. \end{aligned}$$

So, by induction on i, we see that Eq. (1.1) holds for all $i \geq 1$ and $j = 1$. Assuming Eq. (1.1) holds for some $j > 1$ and all $i \geq 1$, an analogous argument (using induction on i) shows that it also holds for $j+1$ and all $i \geq 1$. So, by induction on j, we are done. Conversely, notice that any specification of the values on the axes defines a point in X via Eq. (1.1).

Recall that the cylinder sets form a base for the topology on X. So, to prove the density of periodic points, it is enough to show that any non-empty cylinder set contains a periodic point. Let $[c]_F \subseteq X$ be any such set. Then, since F is finite, there exists an integer $N \geq 1$ such that F is contained in the square $[-N, N]^2$. Let $x \in [c]_F$. We wish to define a periodic point y that matches x on the square $[-N, N]^2$. In light of Eq. (1.1), we define y by making the axes $(x_{i,0})_{i \in \mathbb{Z}}$ and $(x_{0,j})_{j \in \mathbb{Z}}$ of x "$(2N+1)$-periodic". Concretely, we define $y_{i,0} = x_{i+k(2N+1),0}$, where k is the unique integer such that $i + k(2N+1) \in [-N, N]$, for all $i \in \mathbb{Z}$, and $(y_{0,j})_{j \in \mathbb{Z}}$ is defined analogously. By our discussion above, this uniquely defines a point $y \in X$ that matches x on $[-N, N]^2$ and thus lies in $[c]_F$. So it only remains to show that y is in fact periodic. Using Eq. (1.1), we see that

$$\begin{aligned} y_{i,j} &= y_{0,0} + y_{i,0} + y_{0,j} \\ &= y_{0,0} + y_{(i \bmod 2N+1),0} + y_{0,(j \bmod 2N+1)}, \end{aligned}$$

and thus $\sigma_{(m,n)}(y)$ only depends on $(m \bmod 2N+1)$ and $(n \bmod 2N+1)$, which implies that the orbit of y is finite.

1.2 Ergodicity & Mixing

Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Since X is a compact group, we may consider the Haar measure λ_X on X normalized to $\lambda_X(X) = 1$ (see Appendix A). This turns $(X, \mathcal{B}_X, \lambda_X)$ into a probability space, where $\mathcal{B}_X$ denotes the Borel σ-algebra of X. It is readily checked that for any $\gamma \in \Gamma$, the pushforward measure $\lambda'_X(A) := \lambda_X(\alpha_\gamma^{-1}(A))$

is again a Haar measure on X with $\lambda'_X(X) = 1$ and thus equal to λ_X (by uniqueness in Theorem A.3). This shows that α is a measure-preserving action on a probability space, entering the realm of Ergodic Theory.[8]

Definition 1.15. A measurable subset $A \subseteq X$ is called *almost α-invariant* if we have $\lambda_X(A \mathbin{\triangle} \alpha_\gamma^{-1}(A)) = 0$ for every $\gamma \in \Gamma$, where $\triangle$ denotes the symmetric difference.

Definition 1.16. We say that the action α is *ergodic* if any almost α-invariant measurable subset $A \subseteq X$ satisfies $\lambda_X(A) \in \{0,1\}$.

We can think of ergodicity as an irreducibility condition from a measurable dynamics perspective: We cannot find a proper (measure < 1), non-trivial (measure > 0) subsystem. The following lemma makes this observation a bit more concrete.

Lemma 1.17. *For any $A \in \mathcal{B}_X$ the following are equivalent:*

(a) The set A is almost α-invariant.
(b) There exists an $A' \in \mathcal{B}_X$ such that $\lambda_X(A \mathbin{\triangle} A') = 0$ and A' is α-invariant.

So, if α were not ergodic, then using Lemma 1.17 we can find a measurable subset $Y \subseteq X$ that is α-invariant and satisfies $\lambda_X(Y) \in (0,1)$. That is, we may find a subsystem that is proper and non-trivial in a measure-theoretic sense.

Proof of Lemma 1.17. (a) $\Longrightarrow$ (b): Let $A \in \mathcal{B}_X$ be an almost α-invariant subset. We define the set $A' = \bigcup_{\gamma \in \Gamma} \alpha_\gamma^{-1}(A)$. Then A' is α-invariant and

$$\lambda_X(A \mathbin{\triangle} A') = \lambda_X(A' \setminus A) = \lambda_X\left(\bigcup_{\gamma \in \Gamma} \alpha_\gamma^{-1}(A) \setminus A\right) = 0,$$

by σ-additivity of λ_X.

(b) $\Longrightarrow$ (a): Let $A \in \mathcal{B}_X$ and let $A' \in \mathcal{B}_X$ be as in our assumption. Since A' is α-invariant, we have $\alpha_\gamma^{-1}(A') \subseteq A'$ and thus

$$\begin{aligned}\lambda_X(A' \mathbin{\triangle} \alpha_\gamma^{-1}(A')) &= \lambda_X(A' \setminus \alpha_\gamma^{-1}(A')) \\ &= \lambda_X(A') - \lambda_X(\alpha_\gamma^{-1}(A')) \\ &= 0\end{aligned}$$

for all $\gamma \in \Gamma$, where we used that α is measure-preserving. Notice that for any three sets U, V, W, we have $U \mathbin{\triangle} W \subseteq (U \mathbin{\triangle} V) \cup (V \mathbin{\triangle} W)$. Using the above, we conclude

$$\lambda_X(A \mathbin{\triangle} \alpha_\gamma^{-1}(A)) \leq \lambda_X(A \mathbin{\triangle} A') + \lambda_X(A' \mathbin{\triangle} \alpha_\gamma^{-1}(A')) + \lambda_X(\alpha_\gamma^{-1}(A') \mathbin{\triangle} \alpha_\gamma^{-1}(A)) = 0,$$

for any $\gamma \in \Gamma$, as desired. ☐

There are many equivalent characterizations of ergodicity. The following will be particularly useful for our purposes.

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[8] In the spirit of Remark 1.12, we make the following additional observation. If β is an action of a countable monoid M on a compact metrizable group by continuous (not necessarily surjective) endomorphisms, then it is a necessary condition that each β_m, for $m \in M$, is surjective for β to be measure-preserving. In fact, in this case the two are equivalent: A continuous group endomorphism of a compact group is surjective if and only if it preserves a Haar measure.

Proposition 1.18. *For a quasi-algebraic action $\alpha : \Gamma \to \mathrm{Aut}(X)$, the following are equivalent:*

- *(a) The action α is ergodic, i.e., any almost α-invariant measurable subset $A \in \mathcal{B}_X$ has measure $\lambda_X(A) \in \{0, 1\}$.*
- *(b) Any α-invariant measurable subset $A \in \mathcal{B}_X$ satisfies $\lambda_X(A) \in \{0, 1\}$.*
- *(c) Any measurable function $f : X \to \mathbb{C}$ such that $f \circ \alpha_\gamma = f$ holds λ_X-a.e. for every $\gamma \in \Gamma$ is equal to a constant λ_X-a.e.*
- *(d) For any two measurable subsets $A, B \in \mathcal{B}_X$ of positive measure, i.e., $\lambda_X(A)\lambda_X(B) > 0$, there exists a $\gamma \in \Gamma$ such that $\lambda_X(A \cap \alpha_\gamma^{-1}(B)) > 0$.*

Proof. (a) $\iff$ (b): This follows from Lemma 1.17.

(a) $\implies$ (c): Let $f : X \to \mathbb{C}$ be measurable. Then the real and imaginary parts of f are again measurable and α-invariant λ_X-a.e. Thus, we may assume without loss of generality that f is real-valued.

Fix integers $n \geq 1$ and $k \in \mathbb{Z}$ and define

$$A_n^k = \left\{ x \in X : f(x) \in \left[\tfrac{k}{n}, \tfrac{k+1}{n}\right) \right\}.$$

Then $\lambda_X(\alpha_\gamma^{-1}(A_n^k) \triangle A_n^k) = 0$ for every $\gamma \in \Gamma$, by almost α-invariance of f. Our assumption now implies that $\lambda_X(A_n^k) \in \{0, 1\}$. Since $X = \bigsqcup_{k \in \mathbb{Z}} A_n^k$, there must be exactly one $k = k_n$ with $\lambda_X(A_n^{k_n}) = 1$. Then f is constant on the set $\bigcap_{n \geq 1} A_n^{k_n}$, which has measure equal to 1.

(c) $\implies$ (b): Let $A \in \mathcal{B}_X$ be an α-invariant set. Then the characteristic function $\mathbb{1}_A : X \to \mathbb{C}$ of A is measurable and $\mathbb{1}_A \circ \alpha_\gamma = \mathbb{1}_{\alpha_\gamma^{-1}(A)} = \mathbb{1}_A$ holds λ_X-a.e. for all $\gamma \in \Gamma$. Thus, by our assumption, $\mathbb{1}_A$ is constant almost everywhere, i.e., $\lambda_X(A) \in \{0, 1\}$.

(a) $\implies$ (d): Let $A, B \in \mathcal{B}_X$ be measurable subsets of positive measure. Define the α-invariant measurable subset $\tilde{B} = \bigcup_{\gamma \in \Gamma} \alpha_\gamma^{-1}(B)$. Then since B is of positive measure, ergodicity implies $\lambda_X(\tilde{B}) = 1$. We obtain

$$0 < \lambda_X(A) = \lambda_X(A \cap \tilde{B}) \leq \sum_{\gamma \in \Gamma} \lambda_X(A \cap \alpha_\gamma^{-1}(B)),$$

and hence there must exist a $\gamma \in \Gamma$ with $\lambda_X(A \cap \alpha_\gamma^{-1}(B)) > 0$.

(d) $\implies$ (b): Let $A \in \mathcal{B}_X$ be an α-invariant measurable subset. Assume for a contradiction that $0 < \lambda_X(A) < 1$. Then the complement $X \smallsetminus A$ must also be of positive measure and our assumption implies the existence of a $\gamma \in \Gamma$ such that

$$0 < \lambda_X((X \smallsetminus A) \cap \alpha_\gamma^{-1}(A)) \leq \lambda_X((X \smallsetminus A) \cap A) = 0,$$

a contradiction. ☐

Definition 1.19. We call a quasi-algebraic action $\alpha : \Gamma \to \mathrm{Aut}(X)$ *(strongly) mixing* if we have

$$\lim_{\gamma \to \infty} \lambda_X(A \cap \alpha_\gamma^{-1}(B)) = \lambda_X(A)\lambda_X(B),$$

for any two measurable subsets $A, B \in \mathcal{B}_X$, where the limit is interpreted as follows: for any sequence $(\gamma_n)_{n \in \mathbb{N}} \subseteq \Gamma$ that eventually leaves every finite subset $F \subseteq \Gamma$, we have $\lim_{n \to \infty} \lambda_X(A \cap \alpha_{\gamma_n}^{-1}(B)) = \lambda_X(A)\lambda_X(B)$.

From a probabilistic perspective, a mixing action is an action where any two events are asymptotically independent as they "move apart" in Γ. The action effectively "mixes" the space so that any dependence between the two events is eliminated.

Lemma 1.20. *If a quasi-algebraic action $\alpha : \Gamma \to \operatorname{Aut}(X)$ is mixing, then it is also ergodic.*

Proof. Since Γ is infinite, we may choose a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ that eventually leaves every finite subset. Let $A \in \mathcal{B}_X$ be an α-invariant set. Plugging it into the definition of a mixing system along this sequence, we obtain $\lambda_X(A) = \lambda_X(A)^2$, hence $\lambda_X(A) \in \{0, 1\}$. By Proposition 1.18, we are done. $\square$

Proposition 1.21. *Let $\alpha : \Gamma \to \operatorname{Aut}(X)$ be a quasi-algebraic action. Then the following are equivalent:*

(a) *α is mixing.*

(b) *For any $f, g \in L^2(X, \lambda_X)$, we have*

$$\lim_{\gamma\to\infty} \langle f, g \circ \alpha_\gamma \rangle = \int_X f \, d\lambda_X \overline{\int_X g \, d\lambda_X}.$$

(c) *For any $f, g \in L_0^2(X, \lambda_X)$, we have*

$$\lim_{\gamma\to\infty} \langle f, g \circ \alpha_\gamma \rangle = 0,$$

where $L_0^2(X, \lambda_X) = \left\{h \in L^2(X, \lambda_X) : \int_X h \, d\lambda_X = 0\right\}$.

Proof. (b) $\Longrightarrow$ (a): Let $A, B \in \mathcal{B}_X$ be measurable subsets of X. Consider the characteristic functions $\mathbb{1}_A$ and $\mathbb{1}_B$ of A and B respectively. We compute the inner product

$$\begin{aligned}
\langle \mathbb{1}_A, \mathbb{1}_B \circ \alpha_\gamma \rangle &= \int_X \mathbb{1}_A(x) \overline{\mathbb{1}_B(\alpha_\gamma(x))} \, d\lambda_X(x) \\
&= \int_X \mathbb{1}_A(x) \mathbb{1}_{\alpha_\gamma^{-1}(B)}(x) \, d\lambda_X(x) \\
&= \int_X \mathbb{1}_{A\cap\alpha_\gamma^{-1}(B)}(x) \, d\lambda_X(x) \\
&= \lambda_X(A \cap \alpha_\gamma^{-1}(B)).
\end{aligned}$$

On the other hand, we have

$$\int_X \mathbb{1}_A \, d\lambda_X \overline{\int_X \mathbb{1}_B \, d\lambda_X} = \lambda_X(A)\lambda_X(B).$$

Thus, our assumption implies $\lim_{\gamma\to\infty} \lambda_X(A \cap \alpha_\gamma^{-1}(B)) = \lambda_X(A)\lambda_X(B)$, as desired.

(a) $\Longrightarrow$ (b): The computation above and our assumption that α is mixing show that (b) holds for characteristic functions. By sesquilinearity of the inner product and linearity of the integral, we see that (b) holds for all simple functions (finite linear combinations of characteristic functions).

For the general case, let $f, g \in L^2(X, \lambda_X)$ and fix $\varepsilon > 0$. Recall that the set of all simple functions is dense in $L^2(X, \lambda_X)$. Therefore, we may choose simple functions ϕ and ψ such that $\|f - \phi\|_2 < \varepsilon$ and $\|g - \psi\|_2 < \varepsilon$. We estimate

$$\begin{aligned}
|\langle f, g \circ \alpha_\gamma \rangle - \langle f, \mathbb{1}_X \rangle \langle \mathbb{1}_X, g \rangle| \le{} & |\langle f, g \circ \alpha_\gamma \rangle - \langle \phi, \psi \circ \alpha_\gamma \rangle| \\
& + |\langle \phi, \psi \circ \alpha_\gamma \rangle - \langle \phi, \mathbb{1}_X \rangle \langle \mathbb{1}_X, \psi \rangle| \\
& + |\langle \phi, \mathbb{1}_X \rangle \langle \mathbb{1}_X, \psi \rangle - \langle f, \mathbb{1}_X \rangle \langle \mathbb{1}_X, g \rangle|,
\end{aligned} \tag{1.2}$$

for any $\gamma \in \Gamma$. Note that since α_γ is measure preserving, $U_\gamma : h \mapsto h \circ \alpha_\gamma$ defines a unitary operator on $L^2(X, \lambda_X)$, for all $\gamma \in \Gamma$. Using this and the Cauchy–Schwarz inequality, we can further estimate the first term of Eq. (1.2):

$$\begin{aligned} |\langle f, g \circ \alpha_\gamma\rangle - \langle \phi, \psi \circ \alpha_\gamma\rangle| &= |\langle f - \phi, g \circ \alpha_\gamma\rangle + \langle \phi, (g - \psi) \circ \alpha_\gamma\rangle| \\ &\leq \varepsilon(\|g\|_2 + \|\phi\|_2) \\ &\leq \varepsilon(\|g\|_2 + \|f\|_2 + \varepsilon). \end{aligned}$$

For the third term of Eq. (1.2), we find

$$\begin{aligned} |\langle \phi, \mathbb{1}_X\rangle\langle \mathbb{1}_X, \psi\rangle - \langle f, \mathbb{1}_X\rangle\langle \mathbb{1}_X, g\rangle| &= |\langle \phi - f, \mathbb{1}_X\rangle\langle \mathbb{1}_X, \psi\rangle + \langle f, \mathbb{1}_X\rangle\langle \mathbb{1}_X, \psi - g\rangle| \\ &\leq \varepsilon(\|\psi\|_2 + \|f\|_2) \\ &\leq \varepsilon(\|g\|_2 + \|f\|_2 + \varepsilon). \end{aligned}$$

Finally, since ϕ and ψ are simple functions, the second term of Eq. (1.2) goes to zero as $\gamma \to \infty$. We thus obtain

$$\limsup_{\gamma\to\infty} |\langle f, g \circ \alpha_\gamma\rangle - \langle f, \mathbb{1}_X\rangle\langle \mathbb{1}_X, g\rangle| \leq 2\varepsilon(\|f\|_2 + \|g\|_2 + \varepsilon),$$

for any $\varepsilon > 0$, which implies (b).

(b) $\Longrightarrow$ (c): If $f, g \in L^2_0(X, \lambda_X)$, then by definition $\int_X f \, d\lambda_X = \int_X g \, d\lambda_X = 0$. Plugging this into the identity in (b) immediately yields $\lim_{\gamma\to\infty}\langle f, g \circ \alpha_\gamma\rangle = 0$.

(c) $\Longrightarrow$ (b): Let $f, g \in L^2(X, \lambda_X)$. We decompose both functions into their constant parts and their zero-mean parts: $f = c_f \mathbb{1}_X + f_0$ and $g = c_g \mathbb{1}_X + g_0$, where $c_f = \int_X f \, d\lambda_X$, $c_g = \int_X g \, d\lambda_X$, and $f_0, g_0 \in L^2_0(X, \lambda_X)$. Using sesquilinearity of the inner product, we compute

$$\begin{aligned} \langle f, g \circ \alpha_\gamma\rangle &= \langle c_f \mathbb{1}_X + f_0, (c_g \mathbb{1}_X + g_0) \circ \alpha_\gamma\rangle \\ &= c_f\overline{c_g}\langle \mathbb{1}_X, \mathbb{1}_X\rangle + c_f\langle \mathbb{1}_X, g_0 \circ \alpha_\gamma\rangle + \overline{c_g}\langle f_0, \mathbb{1}_X\rangle + \langle f_0, g_0 \circ \alpha_\gamma\rangle. \end{aligned}$$

Since $\lambda_X(X) = 1$, we have $\langle \mathbb{1}_X, \mathbb{1}_X\rangle = 1$. Because $f_0 \in L^2_0(X, \lambda_X)$, the term $\langle f_0, \mathbb{1}_X\rangle = \int_X f_0 \, d\lambda_X$ vanishes. Furthermore, since α_γ preserves λ_X, the function $g_0 \circ \alpha_\gamma$ also has mean zero, meaning $\langle \mathbb{1}_X, g_0 \circ \alpha_\gamma\rangle = 0$. We are thus left with

$$\langle f, g \circ \alpha_\gamma\rangle = c_f\overline{c_g} + \langle f_0, g_0 \circ \alpha_\gamma\rangle.$$

Taking the limit as $\gamma \to \infty$, our assumption (c) implies that the second term vanishes, yielding $\lim_{\gamma\to\infty}\langle f, g \circ \alpha_\gamma\rangle = c_f\overline{c_g} = \int_X f \, d\lambda_X \overline{\int_X g \, d\lambda_X}$, which establishes (b). ☐

Example 1.22 (Arnold's Cat Map III)**.** We return to Arnold's Cat Map $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ discussed in Examples 0.1 and 1.9. From what we have seen so far, T_A exhibits chaotic behavior. Fig. 3, in particular, intuitively suggests that T_A might be mixing. This is indeed the case.

The Peter–Weyl theorem (Theorem B.6) implies that the characters of $\mathbb{T}^2$ form a Hilbert basis of $L^2(\mathbb{T}^2, \lambda_{\mathbb{T}^2})$. Thus, by an approximation argument analogous to the one in the proof of Proposition 1.21, it is enough to show that part (b) of Proposition 1.21 holds for all characters of $\mathbb{T}^2$. By combining Propositions A.8 and A.10, we know that the characters are given by $\chi_{\mathbf{n}} : \mathbb{T}^2 \to \mathbb{S}^1, \mathbf{x} \mapsto \exp(2\pi i\langle \mathbf{x}, \mathbf{n}\rangle)$ for $\mathbf{n} \in \mathbb{Z}^2$. Thus, we wish to show

$$\lim_{k\to\infty}\langle \chi_{\mathbf{n}}, \chi_{\mathbf{m}} \circ T_A^k\rangle = \int_{\mathbb{T}^2} \chi_{\mathbf{n}} \, d\lambda_{\mathbb{T}^2} \overline{\int_{\mathbb{T}^2} \chi_{\mathbf{m}} \, d\lambda_{\mathbb{T}^2}}, \tag{1.3}$$

for any $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^2$ (and $k \to \infty$ in $\mathbb{Z}$).

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. Using Fubini's theorem, we compute

$$\begin{aligned}
\int_{\mathbb{T}^2} \chi_{\mathbf{n}}(\mathbf{x})\, d\lambda_{\mathbb{T}^2}(\mathbf{x}) &= \int_{\mathbb{T}^2} \exp(2\pi i \langle \mathbf{x}, \mathbf{n}\rangle)\, d\lambda_{\mathbb{T}^2}(\mathbf{x}) \\
&= \int_0^1 \int_0^1 \exp(2\pi i (x_1 n_1 + x_2 n_2))\, dx_2\, dx_1 \\
&= \left(\int_0^1 \exp(2\pi i x_1 n_1)\, dx_1\right) \left(\int_0^1 \exp(2\pi i x_2 n_2)\, dx_2\right) \\
&= \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Given $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$, we further compute

$$\begin{aligned}
\langle \chi_{\mathbf{n}}, \chi_{\mathbf{m}} \circ T_A^k \rangle &= \int_{\mathbb{T}^2} \exp(2\pi i \langle \mathbf{x}, \mathbf{n}\rangle) \overline{\exp(2\pi i \langle A^k \mathbf{x}, \mathbf{m}\rangle)}\, d\lambda_{\mathbb{T}^2}(\mathbf{x}) \\
&= \int_{\mathbb{T}^2} \exp(2\pi i \langle \mathbf{x}, \mathbf{n}\rangle) \exp(2\pi i \langle \mathbf{x}, -(A^T)^k \mathbf{m}\rangle)\, d\lambda_{\mathbb{T}^2}(\mathbf{x}) \\
&= \int_{\mathbb{T}^2} \exp(2\pi i \langle \mathbf{x}, \mathbf{n} - (A^T)^k \mathbf{m}\rangle)\, d\lambda_{\mathbb{T}^2}(\mathbf{x}) \\
&= \int_{\mathbb{T}^2} \chi_{\mathbf{n} - (A^T)^k \mathbf{m}}\, d\lambda_{\mathbb{T}^2}.
\end{aligned}$$

We are now ready to conclude. If $\mathbf{m} = \mathbf{0}$, then, by the above calculations, the left- and right-hand sides of Eq. (1.3) are equal for all $k \in \mathbb{Z}$. Otherwise, if $\mathbf{m} \neq \mathbf{0}$, then the right-hand side of Eq. (1.3) is equal to 0. Thus, it is enough to show that $\mathbf{n} - (A^T)^k \mathbf{m} = \mathbf{0}$ can hold for at most one $k \in \mathbb{Z}$. Indeed, assume $(A^T)^k \mathbf{m} = (A^T)^l \mathbf{m}$ for some distinct $k, l \in \mathbb{Z}$. Then we obtain $(A^T)^{k-l} \mathbf{m} = \mathbf{m}$ with $k - l \neq 0$, which implies that A^T has a root of unity as an eigenvalue, a contradiction. We thus obtain the desired convergence in all cases, proving that T_A is mixing.

Remark 1.23. Note that the proof that Arnold's Cat Map T_A is mixing in Example 1.22 boiled down to the fact that the matrix A did not have any roots of unity as eigenvalues. Indeed, similar to Example 1.9 and Remark 1.10, the following holds. Let $T_B : \mathbb{T}^d \to \mathbb{T}^d$ be a toral automorphism given by matrix multiplication with some $B \in \mathrm{GL}_d(\mathbb{Z})$. If B has no roots of unity as eigenvalues, then T_B is mixing.

Example 1.24 (Toral Rotation III)**.** We revisit the dynamical system from Examples 0.2 and 1.11. Let $T_R : \mathbb{T}^2 \to \mathbb{T}^2$ be given by matrix multiplication with $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let r_1, r_2 be such that $0 < r_1 < r_2 < 1/2$ and define $Y \subseteq \mathbb{T}^2$ as the image of

$$Y' = \{x \in \mathbb{R}^2 : r_1 \leq \|x\| \leq r_2\}$$

under the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. Then it follows that Y is invariant under T_R and $\lambda_{\mathbb{T}^2}(Y) = (r_2^2 - r_1^2)\pi$ (see Fig. 1.2), proving that T_R is not ergodic and thus also not mixing (by Lemma 1.20). In view of Remark 1.23, we mention that R has eigenvalues $\pm i \in \mathbb{C}$.

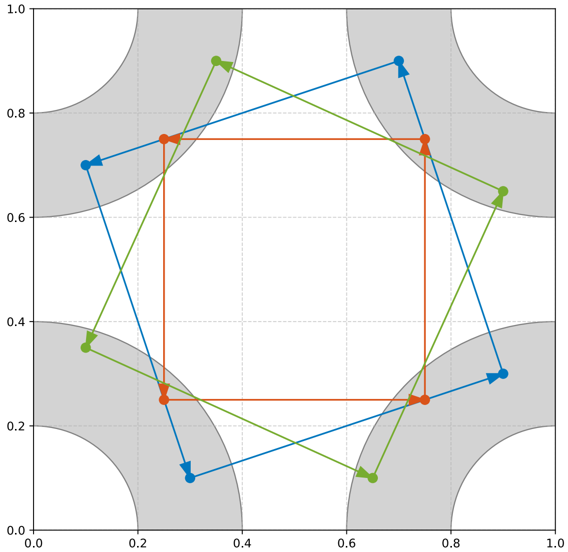

Figure 1.2: Highlighted is the subset Y invariant under $T_R : \mathbb{T}^2 \to \mathbb{T}^2$ and three orbits of points in Y.

Example 1.25 (Times Two Map III)**.** As discussed in Remark 1.12, all of the material developed in this section is also applicable to the Times Two Map $T_2 : \mathbb{T} \to \mathbb{T}$. Our goal is to prove that T_2 is mixing. Using Fourier analysis and imitating the argument presented in Example 1.22 would yield a concise proof. However, we wish to present a, possibly more illuminating, geometrical argument.

For a subset A of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we define

$$\frac{A}{2} = \left\{ \tfrac{r}{2} + \mathbb{Z} : r \in [0,1) \text{ and } r + \mathbb{Z} \in A \right\} \subseteq \mathbb{T}.$$

Notice that this is simply notation for our intuition of shrinking $A \subseteq [0,1) + \mathbb{Z}$ by half to $\frac{A}{2} \subseteq [0, 1/2) + \mathbb{Z}$. It is readily checked that

$$T_2^{-1}(A) = \frac{A}{2} \sqcup \left(\frac{A}{2} + \frac{1}{2} \right) \quad \text{and} \quad \lambda_{\mathbb{T}}\left(\frac{A}{2} \right) = \frac{\lambda_{\mathbb{T}}(A)}{2}$$

holds for all measurable subsets A of $\mathbb{T}$, where we use $\sqcup$ to indicate that the union is disjoint. We therefore see that taking the pre-image of A under T_2 corresponds to shrinking A by half and placing one such copy in $[0, \frac{1}{2}) + \mathbb{Z}$ and another in $[\frac{1}{2}, 1) + \mathbb{Z}$. Thus, $T_2^{-k}(A)$ is made up of 2^k copies of A shrunken by a factor of 2^{-k} and placed in the dyadic intervals $[\frac{j}{2^k}, \frac{j+1}{2^k})$ for $j = 0, \ldots, 2^k - 1$ (see Fig. 1.3 for an illustration). To make this more precise, we may analogously define $\frac{A}{2^k}$ for any integer $k \geq 1$ and conclude

$$T_2^{-k}(A) = \bigsqcup_{j=0}^{2^k-1} \left(\frac{A}{2^k} + \frac{j}{2^k} \right) \quad \text{and} \quad \lambda_{\mathbb{T}}\left(\frac{A}{2^k} \right) = \frac{\lambda_{\mathbb{T}}(A)}{2^k},$$

by induction on k. We further observe that $\frac{A}{2^k} + \frac{j}{2^k}$ is contained in $[\frac{j}{2^k}, \frac{j+1}{2^k}) + \mathbb{Z}$ for every $j = 0, \ldots, 2^k - 1$.

Now, given a measurable subset $A \subseteq \mathbb{T}$ and a dyadic half-open interval B, i.e., $B = [\frac{j}{2^k}, \frac{j+1}{2^k}) + \mathbb{Z}$ for integers $k \geq 0$ and $j \in \{0, \ldots, 2^k - 1\}$, it follows from the observations above that

$$\lambda_{\mathbb{T}}(T_2^{-n}(A) \cap B) = \lambda_{\mathbb{T}}(A)\lambda_{\mathbb{T}}(B),$$

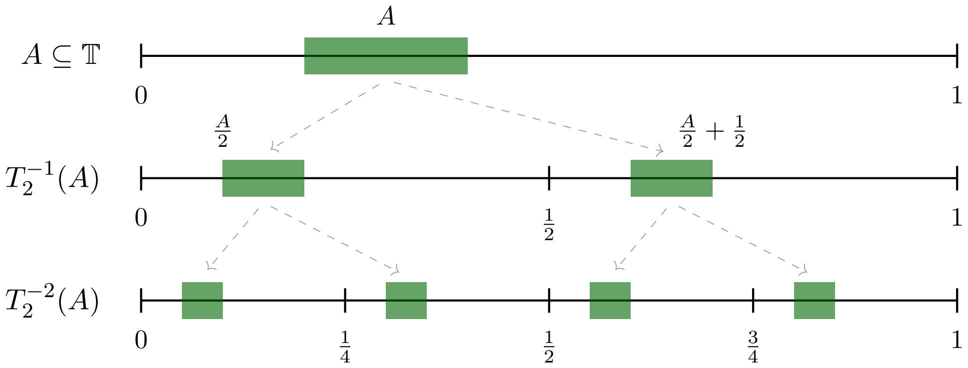

Figure 1.3: Taking pre-images under the Times Two Map.

for all $n \geq k$ (see Fig. 1.4 for an illustration). Notice that this shows the mixing property from Definition 1.19 given that one of the sets is a dyadic interval. Since the dyadic intervals generate the Borel σ-algebra of $\mathbb{T}$, this extends to arbitrary measurable subsets by an approximation argument, thus proving that T_2 is mixing.

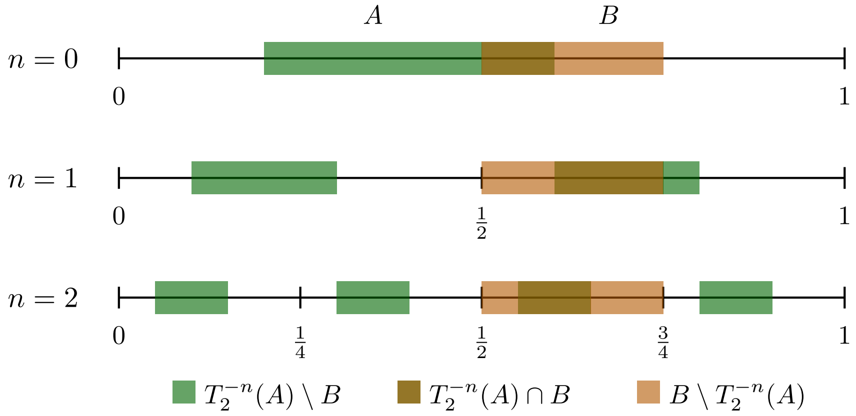

Figure 1.4: An illustration of the mixing property for the Times Two Map with $A = [1/5, 3/5]$ and $B = [2/4, 3/4]$.

Example 1.26 (Square Shift III)**.** We revisit the symbolic shift $\sigma : \mathbb{Z}^2 \to \mathrm{Aut}(X)$ discussed in Examples 0.4 and 1.14.[9] Our goal is to prove that σ is ergodic but not mixing, showing that, in general, the notion of mixing is in fact a stronger requirement than ergodicity.

Since the set of continuous functions is dense in $L^2(X, \lambda_X)$, it is enough to verify part (c) of Proposition 1.18 for continuous functions to conclude that σ is ergodic. Let d_X be the compatible metric on X defined in Example 1.14 and let $f : X \to \mathbb{C}$ be a continuous σ-invariant function. We will prove that for any two points $x, y \in X$ and any $\varepsilon > 0$, we have $|f(x) - f(y)| < \varepsilon$, which implies that f is constant. Since X is compact, f is uniformly continuous and there exists a $\delta > 0$ such that $|f(z) - f(\tilde{z})| < \varepsilon/2$ whenever $d_X(z, \tilde{z}) < \delta$.

The idea is now to find a point in X that is close to x or close to y depending on how we "center" it via application of σ. To do so, recall that given two finite admissible patterns, we may glue them together diagonally to create a new finite admissible pattern (we saw this

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[9] The descriptor *symbolic* refers to the fact that the underlying alphabet, here $\mathbb{Z}/2\mathbb{Z}$, is finite and discrete.

in Example 1.14). With this in mind, we define

$$c_{\mathbf{n}} = \begin{cases} x_{\mathbf{n}} & \text{if } \mathbf{n} \in [-N,N]^2 \\ y_{\mathbf{n}} & \text{if } \mathbf{n} \in [2N,4N]^2 \end{cases}$$

for $\mathbf{n} \in ([-N,N]^2 \cup [2N,4N]^2) \cap \mathbb{Z}^2$ and extend it to a point $c \in X$, where N is chosen large enough such that $d_X(x,c) < \delta$ and $d_X(\sigma_{(3N,3N)}(y), \sigma_{(3N,3N)}(c)) < \delta$. Now using σ-invariance of f, we obtain

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(\sigma_{(3N,3N)}(y)) - f(\sigma_{(3N,3N)}(c))| < \varepsilon,$$

which proves that σ is ergodic.

To show that σ is not mixing, we will explicitly construct two measurable subsets that do not satisfy the condition in Definition 1.19. But first, we need to better understand the Haar measure λ_X on X. Recall that the cylinder sets form a countable base for the topology on X and thus generate the σ-algebra $\mathcal{B}_X$. So the Haar measure is determined by its values on cylinder sets. Using this, one may check that setting $\lambda_X([c]_F) = |\pi_F(X)|^{-1}$, for any non-empty finite subset $F \subseteq \mathbb{Z}^2$ and any $c \in \pi_F(X)$, does indeed define the Haar measure (as usual, $\pi_F : (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} \to (\mathbb{Z}/2\mathbb{Z})^F$ denotes the canonical projection).

We define the sets

$$\begin{aligned} A &= \{(x_{\mathbf{n}}) \in X : x_{(0,0)} + x_{(1,0)} = 0\} \\ B &= \{(x_{\mathbf{n}}) \in X : x_{(0,0)} + x_{(1,0)} = 1\}, \end{aligned}$$

and notice that by our above discussion, we have $\lambda_X(A) = \lambda_X(B) = 1/2$. Furthermore, we observe that $\sigma_{(0,n)}(B) = \{(x_{\mathbf{n}}) \in X : x_{(0,-n)} + x_{(1,-n)} = 1\}$. Using Eq. (1.1), we see that the condition $x_{(0,-n)} + x_{(1,-n)} = 1$ is equivalent to the condition $x_{(0,0)} + x_{(1,0)} = 1$. Thus, we have $\lambda_X(A \cap \sigma_{(0,n)}(B)) = 0 \neq \lambda_X(A)\lambda_X(B)$ for all integers n, proving that σ is not mixing.

1.3 Expansiveness

Definition 1.27. We call a quasi-algebraic action $\alpha : \Gamma \to \operatorname{Aut}(X)$ *expansive* if there exists a neighborhood N of the identity 1_X in X such that

$$\bigcap_{\gamma \in \Gamma} \alpha_\gamma^{-1}(N) = \{1_X\}.$$

In this case, we also call N an *expansive neighborhood* of the identity.

There is a useful equivalent formulation of expansiveness in our context.

Lemma 1.28. *Let $\alpha : \Gamma \to \operatorname{Aut}(X)$ be a quasi-algebraic action. Then α is expansive if and only if for any compatible left-invariant metric d_X on X, there exists a constant $\delta > 0$ such that for any two distinct points $x, y \in X$ there exists a $\gamma \in \Gamma$ such that $d_X(\alpha_\gamma(x), \alpha_\gamma(y)) \geq \delta$.*

Proof. Fix a compatible left-invariant metric d_X on X, which exists by Theorem A.2.

Assume first that α is expansive with an expansive neighborhood N of the identity 1_X. Then there exists a $\delta > 0$ such that the open ball around the identity $B_\delta(1_X)$ is contained in N. Let $x, y \in X$ be any two distinct points. Then $y^{-1}x \neq 1_X$ and, by definition of expansiveness, there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma(y^{-1}x) \notin N$. Thus, $\alpha_\gamma(x) \notin \alpha_\gamma(y)B_\delta(1_X) = B_\delta(\alpha_\gamma(y))$, i.e., $d_X(\alpha_\gamma(x), \alpha_\gamma(y)) \geq \delta$, as desired.

Conversely, assume that there exists a $\delta > 0$ as in the statement of the lemma. We set $N = B_\delta(1_X)$ and let $x \in X \setminus \{1_X\}$. Then, by assumption, there exists a $\gamma \in \Gamma$ such that $d_X(\alpha_\gamma(x), \alpha_\gamma(1_X)) \geq \delta$. This implies $\alpha_\gamma(x) \notin N$, i.e., $x \notin \alpha_\gamma^{-1}(N)$. Since x was arbitrary in $X \setminus \{1_X\}$, we may conclude $\bigcap_{\gamma \in \Gamma} \alpha_\gamma^{-1}(N) = \{1_X\}$, as desired. □

The above makes it apparent that expansiveness formalizes the idea of a system being sensitive to initial conditions. Any two distinct points, no matter how close they are to each other, will eventually become separated by a distance of at least δ. In this sense, expansiveness is a chaotic property.

The following lemma will be useful to us later in our discussion.

Lemma 1.29. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action and $Y \subseteq X$ a closed, normal, and α-invariant subgroup. If α^Y and $\alpha^{X/Y}$ are expansive, then so is α.*

Proof. Let N_Y and $N_{X/Y}$ be expansive neighborhoods of the identity in Y and X/Y, respectively. We may assume that both N_Y and $N_{X/Y}$ are open neighborhoods. Let $U \subseteq X$ be open such that $N_Y = U \cap Y$ and let $V = \pi^{-1}(N_{X/Y})$, where $\pi : X \to X/Y$ denotes the canonical projection. We claim that $N = U \cap V$ is an expansive neighborhood of the identity in X. Notice that it is enough to show that for any $x \in N \setminus \{1_X\}$, there is a $\gamma \in \Gamma$ such that $\alpha_\gamma(x) \notin N$. If $x \in N \setminus \{1_X\}$ lies in Y, then it also lies in $N_Y \setminus \{1_X\}$ and thus there is a $\gamma \in \Gamma$ such that $\alpha_\gamma^Y(x) = \alpha_\gamma(x) \notin N_Y$. Since Y is α-invariant, $\alpha_\gamma(x)$ still lies in Y and thus $\alpha_\gamma(x) \notin U \supseteq N$. If $x \in N \setminus \{1_X\}$ does not lie in Y, then $\pi(x)$ is not the identity and there exists a $\gamma \in \Gamma$ such that $\alpha_\gamma^{X/Y}(\pi(x)) \notin N_{X/Y}$. By definition, the projection π and the action α commute and thus $\pi(\alpha_\gamma(x)) \notin N_{X/Y}$. This yields $\alpha_\gamma(x) \notin V$, which implies $\alpha_\gamma(x) \notin N$, and thus concludes the proof. □

We invite the reader to go back to our running examples (Arnold's Cat Map, Toral Rotation, Times Two Map, and Square Shift) and think about whether or not these systems are expansive before continuing to read.

Example 1.30 (Arnold's Cat Map IV)**.** We revisit Arnold's Cat Map $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ previously discussed in Examples 0.1, 1.9 and 1.22. We invite the reader to go back and take another look at Fig. 2. With the characterization from Lemma 1.28 in mind, Fig. 2 suggests that T_A is expansive.

Let $\|A\|$ denote the operator norm of A and note that we have $\|Ax\| \leq \|A\|\|x\|$ and $\|A^{-1}x\| \leq \|A^{-1}\|\|x\| \leq \|A\|\|x\|$, where the latter relies on the fact that $|\det A| = 1$.[10] We choose $\delta > 0$ such that $\|A\|\delta < 1/2$ and claim that $\pi(B_\delta(0))$ is an expansive neighborhood of the identity in $\mathbb{T}^2$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denotes the canonical projection. For a contradiction, assume that there is a non-zero $x \in \mathbb{T}^2$ such that $T_A^n(x) \in \pi(B_\delta(0))$ for all $n \in \mathbb{Z}$. Then there exists a unique $\tilde{x} \in B_\delta(0)$ such that $\pi(\tilde{x}) = x$ and $\pi(A^n\tilde{x}) \in \pi(B_\delta(0))$ for all $n \in \mathbb{Z}$. By our choice of δ, we have $\|A\tilde{x}\| < 1/2$ and $\|A^{-1}\tilde{x}\| < 1/2$. Combining this with $\pi(A^{\pm 1}\tilde{x}) \in \pi(B_\delta(0))$ yields $A^{\pm 1}\tilde{x} \in B_\delta(0)$ and by induction $A^n\tilde{x} \in B_\delta(0)$ for all $n \in \mathbb{Z}$. However, since A has no eigenvalues of modulus 1, we can decompose $\tilde{x} \neq 0$ into its components in the stable and unstable eigenspaces E^s and E^u (as defined in Example 1.9) and see that either $\|A^n\tilde{x}\|$ or $\|A^{-n}\tilde{x}\|$ goes to infinity as $n \to \infty$, a contradiction. This proves that T_A is expansive.

Remark 1.31. We note that the proof of expansiveness of Arnold's Cat Map T_A in Example 1.30 above only really relied on the fact that A has no eigenvalues of modulus 1. Indeed, similar to Remarks 1.10 and 1.23, the following holds. Let $T_B : \mathbb{T}^d \to \mathbb{T}^d$ be a toral automorphism given by matrix multiplication with $B \in \mathrm{GL}_d(\mathbb{Z})$. If B has no eigenvalues of modulus 1, then T_B is expansive.

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[10] The general form of this inequality for some $B \in \mathrm{GL}_d(\mathbb{Z})$ is $\|B^{-1}\| \leq \|B\|^{d-1}$ (this can be seen by considering the singular value decomposition of B). Thus, for the toral automorphism $T_B : \mathbb{T}^d \to \mathbb{T}^d$ induced by B, we would choose $\delta < \frac{1}{2\|B\|^{d-1}}$ to prove expansiveness (assuming B has no eigenvalues of modulus 1).

Example 1.32 (Toral Rotation IV)**.** We return to the Toral Rotation $T_R : \mathbb{T}^2 \to \mathbb{T}^2$ from Examples 0.2, 1.11 and 1.24. Recalling that $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we see that $\|Rv\| = \|v\|$ for all $v \in \mathbb{R}^2$. That is, R is an isometry on $\mathbb{R}^2$. We thus obtain

$$\begin{aligned} d_{\mathbb{T}^2}(T_R(x), T_R(y)) &= \min_{k \in \mathbb{Z}^2} \|R\tilde{x} - R\tilde{y} - k\| \\ &= \min_{k \in \mathbb{Z}^2} \|R\tilde{x} - R\tilde{y} - Rk\| \\ &= \min_{k \in \mathbb{Z}^2} \|\tilde{x} - \tilde{y} - k\| \\ &= d_{\mathbb{T}^2}(x, y), \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are lifts of x and y to $\mathbb{R}^2$, respectively. This shows that T_R is an isometry and hence that it is not expansive. This confirms our intuition of T_R being well-behaved and expansiveness being a chaotic property.

Example 1.33 (Times Two Map IV)**.** We revisit the Times Two Map $T_2 : \mathbb{T} \to \mathbb{T}$ previously discussed in Examples 0.3, 1.13 and 1.25. Recall that it is not a quasi-algebraic action. Nevertheless, Definition 1.27 and Lemma 1.28 are still applicable to the Times Two Map viewed as an action of the monoid $\mathbb{Z}_{\geq 0}$, and we will show that it is expansive.

We have previously seen that T_2 is topologically transitive as well as mixing. This suggests chaotic behavior and one may thus hypothesize that T_2 is expansive. Indeed, we claim that $B_\delta^{\mathbb{T}}(0)$ for $\delta = 1/4$ is an expansive neighborhood of the identity. Let $x \in \mathbb{T}$ be non-zero. If $x \notin B_\delta^{\mathbb{T}}(0)$, then it is clear that $x \notin \bigcap_{n \geq 0} T_2^{-n}(B_\delta^{\mathbb{T}}(0))$. Otherwise, there exists a unique representative $\tilde{x} \in B_\delta^{\mathbb{R}}(0)$ of x. By our choice of δ, we have $|2\tilde{x}| < 1/2$ and thus

$$d_{\mathbb{T}}(T_2(x), 0) = d_{\mathbb{T}}(2x, 0) = \min_{k \in \mathbb{Z}} |2\tilde{x} - k| = |2\tilde{x}| = 2 \min_{k \in \mathbb{Z}} |\tilde{x} - k| = 2 d_{\mathbb{T}}(x, 0).$$

Now, either we have $2 d_{\mathbb{T}}(x, 0) \geq \delta$, and thus $x \notin T_2^{-1}(B_\delta^{\mathbb{T}}(0))$, or we may repeat the argument above with $T_2(x)$ to obtain $d_{\mathbb{T}}(T_2^2(x), 0) = 2^2 d_{\mathbb{T}}(x, 0)$. Since $d_{\mathbb{T}}(x, 0) > 0$, we will eventually obtain some $n \geq 0$ such that $x \notin T_2^{-n}(B_\delta^{\mathbb{T}}(0))$ and thus $x \notin \bigcap_{n \geq 0} T_2^{-n}(B_\delta^{\mathbb{T}}(0))$. This proves that T_2 is indeed expansive.

Example 1.34 (Square Shift IV)**.** Lastly, we revisit the symbolic shift $\sigma : \mathbb{Z}^2 \to \mathrm{Aut}(X)$ introduced in Example 0.4 and further analyzed in Examples 1.14 and 1.26. We will show that σ is expansive. In fact, the following argument may be used to show that any shift-action over a finite alphabet (in our case $\mathbb{Z}/2\mathbb{Z}$) is expansive.[11]

Consider the following neighborhood of the identity in X given as a cylinder set

$$N = \{(x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in X : x_{\mathbf{0}} = 0\}.$$

Then, for any non-zero $x = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in X$, there exists an $\mathbf{m} \in \mathbb{Z}^2$ such that $x_{\mathbf{m}} \neq 0$, and so $(\sigma_{\mathbf{m}}(x))_{\mathbf{0}} = x_{\mathbf{m}}$. This implies $x \notin \sigma_{\mathbf{m}}^{-1}(N)$, thus proving that σ is expansive.

1.4 The Descending Chain Condition

Definition 1.35. Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. We say that α satisfies the *descending chain condition (d.c.c.)* if for every non-increasing sequence

$$X \supseteq X_1 \supseteq \cdots \supseteq X_k \supseteq \dots$$

of closed α-invariant subgroups of X, there exists an integer $K \geq 1$ such that $X_k = X_K$ for all $k \geq K$.

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[11] We will discuss the notions of shift-actions and subshifts in Section 2.1.

Remark 1.36. We note that if a quasi-algebraic action α satisfies the d.c.c., then it is readily checked that α^V and $\alpha^{X/W}$ also satisfy the d.c.c. for all closed, α-invariant subgroups $V \subseteq X$ and all closed, α-invariant, normal subgroups $W \subseteq X$.

Unlike the properties we have discussed so far, the descending chain condition lacks an intuitive dynamical interpretation and may thus seem somewhat opaque. It is perhaps best thought of as a useful structural property, specifically, a finiteness condition. Indeed, the reader familiar with commutative algebra may compare it to the chain conditions for modules and their finiteness implications.[12] Nevertheless, it does have dynamical consequences. We will see one such consequence in Chapter 3 (specifically, Theorem 3.11) and refer the reader to [Sch95, Section 3] for a more elaborate discussion.

Example 1.37 (Toral Automorphisms and the d.c.c.)**.** Let $B \in \mathrm{GL}_d(\mathbb{Z})$ and consider the induced toral automorphism $T_B : \mathbb{T}^d \to \mathbb{T}^d$. We claim that T_B satisfies the descending chain condition.

We first present an algebraic proof in the spirit of the methods of Chapter 3. Let $H_1 \supseteq H_2 \supseteq \dots$ be a non-increasing chain of closed subgroups of $\mathbb{T}^d$. We pass to the Pontryagin dual by taking annihilators and obtain a non-decreasing chain of closed subgroups $H_1^\perp \subseteq H_2^\perp \subseteq \dots$ in $\widehat{\mathbb{T}^d} \cong \mathbb{Z}^d$. Since $\mathbb{Z}^d$ is a finitely generated module over the Noetherian ring $\mathbb{Z}$, we obtain that $\widehat{\mathbb{T}^d}$ is a Noetherian $\mathbb{Z}$-module. Since each $H_n^\perp$ is a subgroup of the abelian group $\widehat{\mathbb{T}^d}$, it is a $\mathbb{Z}$-submodule. So the Noetherian condition implies that the non-decreasing chain $H_1^\perp \subseteq H_2^\perp \subseteq \dots$ stabilizes, and our original chain stabilizes as well.

Alternatively, one can prove this using the compact Lie group structure of $\mathbb{T}^d$. In fact, *any* quasi-algebraic action on a compact Lie group X satisfies the descending chain condition: by Cartan's closed-subgroup theorem [Lee12, Theorem 20.12], every closed subgroup in the chain $X \supseteq X_1 \supseteq X_2 \supseteq \dots$ is a Lie subgroup. So the sequence of natural numbers given by the dimensions $\dim X_k$ must eventually stabilize. Once $\dim X_k = \dim X_{k+1}$, X_{k+1} must be open in X_k [Lee12, Proposition 5.1], so the identity components must coincide, i.e., $X_{k+1}^\circ = X_k^\circ$, by [Lee12, Proposition 7.15]. One can then conclude by observing that the quotients X_k/X_k° form a non-increasing sequence of finite subgroups and thus stabilize.

The above example shows that Arnold's Cat Map (Example 0.1), the Toral Rotation (Example 0.2), and the Times Two Map (Example 0.3) satisfy the descending chain condition.

Finally, we note that neither argument in Example 1.37 used the invariance of the subgroups under the action. The arguments solely relied on the "smallness" of the underlying space. For the Square Shift (Example 0.4), the same strategy does not work, as the underlying space X is "big". Indeed, one can easily construct a non-increasing sequence of closed subgroups of X that does not stabilize. Nevertheless, the Square Shift does satisfy the descending chain condition and the proof crucially relies on the shift-invariance of the subgroups. We do not provide a proof here as it is an immediate consequence of Theorem 2.8.

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[12] A reader not familiar with commutative algebra is encouraged to think about how one may characterize finite-dimensionality of a vector space using a similar condition on non-increasing sequences of subspaces.

Chapter 2

Rigidity of Quasi-Algebraic Actions

In the preceding chapter, we explored fundamental dynamical properties through the lens of quasi-algebraic actions. However, as previously noted, these notions do not rely on the rich structure of quasi-algebraic actions and are each applicable to a much broader class of dynamical systems. In contrast, the results presented in this chapter rely heavily on the underlying algebraic structure and serve as a bridge from the general dynamical theory of Chapter 1 to the algebraic dictionary we will develop in Chapter 3.

In Section 2.1, we will demonstrate how expansiveness and the descending chain condition each force a quasi-algebraic action to be conjugate to a *Lie subshift*, a class of systems that are particularly well-structured.

In Section 2.2, we turn to the probabilistic notions of ergodicity and mixing. We will use the rich structure of quasi-algebraic actions to deduce spectral characterizations of these properties.

The content of this chapter is based on the results presented in Chapter 1 of Klaus Schmidt's monograph *Dynamical Systems of Algebraic Origin* [Sch95], which were first established in joint work by Bruce Kitchens and Klaus Schmidt in [KS89].

2.1 Lie Subshifts and Markov Type Groups

Definition 2.1. Let Γ be a countable group and G a compact metrizable group. We define the *shift-action* σ of Γ on G^Γ by

$$(\sigma_{\tilde{\gamma}}(x))_\gamma = x_{\gamma\tilde{\gamma}}$$

for all $x = (x_\gamma) \in G^\Gamma$ and $\gamma, \tilde{\gamma} \in \Gamma$. Moreover, if $Y \subseteq G^\Gamma$ is a closed, σ-invariant (i.e., *shift-invariant*) subgroup, then we call $\sigma^Y : \Gamma \to \mathrm{Aut}(Y)$ a *subshift* of G^Γ. If G additionally has the structure of a Lie group, then we call σ^Y a *Lie subshift*.

We note that since Γ is countable and G is compact metrizable, the product G^Γ is again a compact metrizable group (by Tychonoff's theorem and preservation of second-countability under countable products). Furthermore, it is readily checked that σ_γ is a continuous automorphism for all $\gamma \in \Gamma$, and hence that any shift-action is also a quasi-algebraic action. The Square Shift (Example 0.4) is an example of a (Lie) subshift. For more details on why a shift-action is a quasi-algebraic action, we refer to our previous discussion in Example 1.14.

Proposition 2.2. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. If α is expansive, then it is conjugate to a Lie subshift.*

Let us first discuss the idea behind the proof. If α is expansive with an expansive neighborhood $N \subseteq X$, then for any two distinct points, we can find an automorphism α_γ that separates the points "at the scale of N". This implies that the full dynamical trajectory of a point contains enough information to distinguish it from any other trajectory, even when observed at the scale of N. This uniform separation will be the key to obtaining our desired conjugacy.

Proof of Proposition 2.2. Let N be an open expansive neighborhood of the identity 1_X in X and consider the compact subset $K = X \smallsetminus N$. For any $k \in K$, there exists an irreducible unitary representation τ_k of X such that $\tau_k(k) \neq \tau_k(1_X)$ (by Corollary B.8). By continuity of the representation, there exists an open neighborhood U_k of k in X such that $\tau_k(y) \neq \tau_k(1_X)$ for all $y \in U_k$. By compactness of K, there exist finitely many $k_1, \ldots, k_m$ in K such that the sets U_{k_i} cover K. Define the finite-dimensional unitary representation $\tau = \tau_{k_1} \oplus \cdots \oplus \tau_{k_m}$ of X and notice that by construction $\ker \tau \subseteq N$.

Next, set $G = \tau(X)$ and further define the continuous group homomorphism

$$\begin{aligned} \phi : X &\to G^\Gamma \\ x &\mapsto (\tau \circ \alpha_\gamma(x))_{\gamma \in \Gamma}. \end{aligned}$$

For $\tilde{\gamma} \in \Gamma$ and $x \in X$, we have

$$\phi(\alpha_{\tilde{\gamma}}(x)) = (\tau \circ \alpha_{\gamma\tilde{\gamma}}(x))_\gamma = \sigma_{\tilde{\gamma}}((\tau \circ \alpha_\gamma(x))_\gamma) = \sigma_{\tilde{\gamma}}(\phi(x)).$$

Thus, it only remains to show that ϕ is injective. Assume $\phi(x) = \phi(y)$ for some $x, y \in X$. Then $\alpha_\gamma(xy^{-1}) \in \ker \tau$ for all $\gamma \in \Gamma$. Since $\ker \tau \subseteq N$, this implies $xy^{-1} \in \bigcap_{\gamma \in \Gamma} \alpha_\gamma^{-1}(N)$, yielding $x = y$. □

Proposition 2.3. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. If α satisfies the descending chain condition, then it is conjugate to a Lie subshift.*

We will prove Proposition 2.3 via a more general lemma whose proof is very much in the spirit of the proof of Proposition 2.2.

Lemma 2.4. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Then there exists a non-increasing sequence $(V_n)_{n \geq 1}$ of closed, normal, α-invariant subgroups of X such that $\bigcap_{n \geq 1} V_n = \{1_X\}$ and $\alpha^{X/V_n} : \Gamma \to \mathrm{Aut}(X/V_n)$ is conjugate to a Lie subshift for every $n \geq 1$.*

Proof. Since X is metrizable, $L^2(X, \lambda_X)$ is separable. Combining this with the Peter–Weyl theorem (Theorem B.6), we obtain that there are only countably many non-equivalent irreducible unitary representations of X. We collect these in a sequence $(\rho_n)_{n \geq 1}$. For every $n \geq 1$, we define $\tau_n = \rho_1 \oplus \cdots \oplus \rho_n$ and put $G_n = \tau_n(X)$, which is a compact subgroup of the matrix group of some finite-dimensional complex Hilbert space. We further define the continuous group homomorphism

$$\begin{aligned} \phi_n : X &\to G_n^\Gamma \\ x &\mapsto (\tau_n \circ \alpha_\gamma(x))_{\gamma \in \Gamma}. \end{aligned}$$

Then, by construction, we have $\phi_n \circ \alpha_\gamma = \sigma_\gamma \circ \phi_n$ for all $\gamma \in \Gamma$. So $V_n = \ker \phi_n$ is a closed, normal, α-invariant subgroup of X and α^{X/V_n} is conjugate to a Lie subshift via ϕ_n. Since τ_n is a subrepresentation of τ_{n+1}, we obtain that $V_n \supseteq V_{n+1}$ for all $n \geq 1$. Finally, since the irreducible unitary representations of X separate the points of X (by Corollary B.8), we conclude $\bigcap_{n \geq 1} V_n = \{1_X\}$. □

Proof of Proposition 2.3. Let $(V_n)_{n\geq 1}$ be the non-increasing sequence of closed, normal, α-invariant subgroups of X obtained in Lemma 2.4. Since α satisfies the d.c.c., there must exist an $N \geq 1$ such that $V_N = \{1_X\}$, and thus $\alpha^{X/V_N} = \alpha$ is conjugate to a Lie subshift. $\square$

In light of Propositions 2.2 and 2.3, one might suspect a link between expansiveness and the descending chain condition. However, these properties are independent. We have already seen that the Toral Rotation (Example 0.2) satisfies the descending chain condition without being expansive (Examples 1.32 and 1.37). Conversely, the following example demonstrates a quasi-algebraic action that is expansive but fails to satisfy the descending chain condition.

Example 2.5 (An expansive action not satisfying the d.c.c.)**.** Consider the countable abelian group $\Gamma = \bigoplus_{n\geq 2} \mathbb{Z}/n\mathbb{Z}$ as our acting group and the compact Lie group $G = \mathbb{Z}/2\mathbb{Z}$. These induce a shift-action σ of Γ on G^Γ. We first notice that by an argument analogous to that in Example 1.34, the cylinder set

$$N = \{(x_\gamma) \in G^\Gamma : x_{0_\Gamma} = 0\}$$

defines an expansive neighborhood in G^Γ, showing that σ is expansive. We will now prove that σ does not satisfy the descending chain condition. Consider the subgroups $\Gamma_n \subseteq \Gamma$ corresponding to the component $\mathbb{Z}/n\mathbb{Z}$, i.e.,

$$\Gamma_n = \left\{ (\gamma_m) \in \Gamma = \bigoplus_{m\geq 2} \mathbb{Z}/m\mathbb{Z} : \gamma_m = 0 \text{ for all } m \neq n \right\}.$$

Now for every $n \geq 2$, we consider the closed subgroup

$$X_n = \left\{ x \in G^\Gamma : \sum_{\gamma\in\Gamma_m} \sigma_\gamma(x) = 0_{G^\Gamma} \text{ for every } m \leq n \right\}.$$

We observe that these subgroups are also shift-invariant. Indeed, we have

$$\left(\sum_{\gamma\in\Gamma_m} \sigma_\gamma(\sigma_{\gamma_1}(x)) \right)_{\gamma_2} = \sum_{\gamma\in\Gamma_m} x_{\gamma+\gamma_1+\gamma_2} = \left(\sum_{\gamma\in\Gamma_m} \sigma_\gamma(x) \right)_{\gamma_1+\gamma_2},$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and any $x \in G^\Gamma$. We further observe that the sets X_n form a non-increasing sequence. Hence, to conclude that σ does not satisfy the descending chain condition, it is enough to show that the inclusions are strict, i.e., $X_n \smallsetminus X_{n+1} \neq \varnothing$ for all $n \geq 2$. Fix $n \geq 2$ and consider the point $x \in G^\Gamma$ defined by

$$x_\gamma = \begin{cases} 1 & \text{if } \gamma_{n+1} = 0 \text{ and } \gamma_m \in \{0,1\} \text{ for all } m \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

for any $\gamma = (\gamma_m) \in \Gamma$. We claim that $x \in X_n \smallsetminus X_{n+1}$. Let $m \leq n$ and $\eta \in \Gamma$. Then we have

$$\left(\sum_{\gamma\in\Gamma_m} \sigma_\gamma(x) \right)_\eta = \sum_{\gamma\in\Gamma_m} x_{\eta+\gamma}. \tag{2.1}$$

Notice that since $\gamma \in \Gamma_m$, adding γ to η only affects its m-th coordinate. Furthermore, as γ varies in Γ_m, $(\eta+\gamma)_m$ takes each value in $\mathbb{Z}/m\mathbb{Z}$ exactly once. We now distinguish three cases.

If $\eta_{n+1} \neq 0$, then $x_{\eta+\gamma} = 0$ for all $\gamma \in \Gamma_m$ by construction. So the sum in Eq. (2.1) is zero in this case.

Alternatively, if $\eta_{n+1} = 0$ and there exists a $\tilde{m} \leq n$ with $\tilde{m} \neq m$ and $\eta_{\tilde{m}} \notin \{0, 1\}$, then we have by construction that $x_{\eta+\gamma} = 0$ for all $\gamma \in \Gamma_m$, which again shows that the sum in Eq. (2.1) is zero.

Otherwise, we have that $\eta_{n+1} = 0$ and $\eta_{\tilde{m}} \in \{0, 1\}$ for all $\tilde{m} \in \{2, \ldots, n\} \smallsetminus \{m\}$. In this case, $(\eta + \gamma)_m$ takes each value in $\mathbb{Z}/m\mathbb{Z}$ exactly once as γ varies in Γ_m. We thus see that there are exactly two terms equal to 1 in the sum Eq. (2.1) and the other terms are zero, corresponding to $(\eta+\gamma)_m \in \{0, 1\}$ or $(\eta+\gamma)_m \in \mathbb{Z}/m\mathbb{Z} \smallsetminus \{0, 1\}$. Recalling that $G = \mathbb{Z}/2\mathbb{Z}$, we see that the sum in Eq. (2.1) is zero in this case as well.

So we have shown that for any $m \leq n$ and any $\eta \in \Gamma$, the sum in Eq. (2.1) is equal to zero. This implies that $x \in X_n$. To see that $x \notin X_{n+1}$, consider Eq. (2.1) for $m = n+1$ and $\eta = 0 \in \Gamma$. In this case, $x_{\eta+\gamma}$ will be equal to 1 exactly once as γ varies in Γ_{n+1}, namely when $\gamma = 0$, so that the sum is non-zero.

In total, we have shown that there exists an $x \in X_n \smallsetminus X_{n+1}$ for any $n \geq 2$ and hence that σ does not satisfy the descending chain condition.

We observe that the group Γ in the example above is not finitely generated. This raises the question of whether the independence of these properties is a general phenomenon or merely an artifact of the group's infinite rank. In other words, does expansiveness imply the descending chain condition if we restrict our attention to more "algebraically small" groups? This motivates the following definition.

Definition 2.6. A countable group Γ is of *Markov type* if the shift-action of Γ on G^Γ satisfies the descending chain condition for every compact Lie group G.

Corollary 2.7. *Let Γ be a Markov type group and $\alpha : \Gamma \to \operatorname{Aut}(X)$ a quasi-algebraic action. If α is expansive, then it satisfies the descending chain condition.*

Proof. This is immediate by combining Propositions 2.2 and 2.3. ☐

For the remainder of this section, we will be concerned with proving that $\mathbb{Z}^d$ belongs to this tame class of Markov type groups for any $d \geq 0$, which will be useful to us in Chapter 3.

Theorem 2.8. *The group $\mathbb{Z}^d$ is of Markov type for all integers $d \geq 0$.*

Remark 2.9. More generally, one can show that the class of Markov type groups includes all polycyclic-by-finite groups [Sch95, Thm 4.2].[1]

Definition 2.10. Let $\sigma : \Gamma \to \operatorname{Aut}(G^\Gamma)$ be a shift-action. A subgroup $X \subseteq G^\Gamma$ is of *finite type* if it is closed, shift-invariant, and there exists a finite subset $F \subseteq \Gamma$ such that

$$X = \{x \in G^\Gamma : \pi_F(\sigma_\gamma(x)) \in \pi_F(X) \text{ for every } \gamma \in \Gamma\},$$

where $\pi_F : G^\Gamma \to G^F$ denotes the canonical projection. In this case, we also say that σ^X is a *subshift of finite type.*

The condition in the above definition may be interpreted as saying that $x \in G^\Gamma$ lies in X if and only if we only observe patterns contained in $\pi_F(X)$ when sliding the finite window F across x. Thus, being contained in X is a local property.

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[1] A polycyclic-by-finite group is a group that has a polycyclic subgroup of finite index. A polycyclic group is a group that admits a subnormal series with cyclic factors. For more on polycyclic-by-finite groups, see [Seg83].

Example 2.11 (Square Shift V)**.** We claim that the Square Shift $\sigma : \mathbb{Z}^2 \to \mathrm{Aut}(X)$ introduced in Example 0.4 is a subshift of finite type. Indeed, recall that the space X is defined as

$$X = \{(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}+(1,1)} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

We can already observe that the elements of X are characterized by the local constraint

$$x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}+(1,1)} = 0 \tag{2.2}$$

for all $\mathbf{n} \in \mathbb{Z}^2$, which suggests that X is of finite type. To make it explicit, set $F = \{(0,0),(1,0),(0,1),(1,1)\} \subseteq \mathbb{Z}^2$ and notice that

$$\pi_F(X) = \{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\} \subseteq (\mathbb{Z}/2\mathbb{Z})^F.$$

Then Eq. (2.2) translates into $\pi_F(\sigma_{\mathbf{n}}(x)) \in \pi_F(X)$.

Proposition 2.12. *Let Γ be a countable group and G a compact Lie group. Then the shift-action $\sigma : \Gamma \to \mathrm{Aut}(G^\Gamma)$ satisfies the descending chain condition if and only if every closed, shift-invariant subgroup of G^Γ is of finite type.*

Proof. Let us first assume that σ satisfies the descending chain condition. Let $X \subseteq G^\Gamma$ be a closed, shift-invariant subgroup. Since Γ is countable, we may choose a sequence $(F_n)_{n\geq 1}$ of increasing finite subsets of Γ with $\bigcup_{n\geq 1} F_n = \Gamma$. Define

$$X_n = \{x \in G^\Gamma : \pi_{F_n}(\sigma_\gamma(x)) \in \pi_{F_n}(X) \text{ for all } \gamma \in \Gamma\},$$

for all $n \geq 1$. We observe that each X_n is by construction shift-invariant, closed by continuity of σ_γ and π_{F_n}, and of finite type. Furthermore, since $F_n \subseteq F_{n+1}$ for all $n \geq 1$, the sets X_n form a non-increasing sequence. Thus, by assumption, there exists an integer $N \geq 1$ such that $X_N = X_n$ for all $n \geq N$.

We claim that $\bigcap_{n\geq 1} X_n = X$, which would imply that $X = X_N$ is of finite type. From the definition of X_n, it is apparent that $X \subseteq X_n$ for all $n \geq 1$, and hence $X \subseteq \bigcap_{n\geq 1} X_n$. For the reverse inclusion, let $x \in \bigcap_{n\geq 1} X_n$. Setting $\gamma = 1_\Gamma$ in the definition, we obtain that $\pi_{F_n}(x) \in \pi_{F_n}(X)$. This implies that for every $n \geq 1$ there exists a $y^{(n)} \in X$ such that $y^{(n)}$ and x agree on F_n. Since the sets F_n exhaust Γ, the sequence $(y^{(n)})_{n\geq 1}$ converges to x in the product topology. Recalling that X was assumed to be closed, we conclude that $x \in X$.

Conversely, assume that any closed, shift-invariant subgroup of G^Γ is of finite type. Let $(X_n)_{n\geq 1}$ be a non-increasing sequence of shift-invariant, closed subgroups. Then we have that $X = \bigcap_{n\geq 1} X_n$ is also a closed, shift-invariant subgroup and thus of finite type. So there exists a finite subset $F \subseteq \Gamma$ such that

$$X = \{x \in G^\Gamma : \pi_F(\sigma_\gamma(x)) \in \pi_F(X) \text{ for all } \gamma \in \Gamma\}.$$

We define

$$Y_n = \{x \in G^\Gamma : \pi_F(\sigma_\gamma(x)) \in \pi_F(X_n) \text{ for all } \gamma \in \Gamma\},$$

and observe that $X \subseteq X_n \subseteq Y_n$. Next, consider the non-increasing sequence $(\pi_F(X_n))_{n\geq 1}$ of closed subgroups in G^F. We claim that $\pi_F(X) = \bigcap_{n\geq 1} \pi_F(X_n)$. Since $X \subseteq X_n$ for all $n \geq 1$, the inclusion $\pi_F(X) \subseteq \bigcap_{n\geq 1} \pi_F(X_n)$ is clear. For the reverse inclusion, let $z \in \bigcap_{n\geq 1} \pi_F(X_n)$. Then there exists an $x^{(n)} \in X_n$ such that $\pi_F(x^{(n)}) = z$ for all $n \geq 1$. Notice that $(x^{(n)})_{n\geq 1}$ defines a sequence in the compact metrizable subspace $X_1 \subseteq G^\Gamma$. Thus, there exists a subsequence $(x^{(n_k)})_{k\geq 1}$ converging to some $x^* \in \bigcap_{n\geq 1} X_n = X$. Now, continuity of the projection implies $\pi_F(x^*) = z \in \pi_F(X)$, proving $\pi_F(X) = \bigcap_{n\geq 1} \pi_F(X_n)$.

Finally, since G^F is a compact Lie group, the non-increasing sequence $(\pi_F(X_n))_{n\geq 1}$ of closed subgroups eventually stabilizes, i.e., there exists an integer $N \geq 1$ such that $\pi_F(X_N) = \pi_F(X_n)$ for all $n \geq N$ (see Example 1.37 for more details). Consequently, we have $\pi_F(X_N) = \pi_F(X)$, which implies $X = Y_N \supseteq X_N \supseteq X_n \supseteq X$ for all $n \geq N$, concluding the proof. $\square$

Before providing the proof of Theorem 2.8, let us briefly discuss the idea behind it. The strategy is to use induction on the dimension d and the characterization of the descending chain condition provided by Proposition 2.12. Thus, our goal will be to prove that any closed, shift-invariant subgroup $V \subseteq G^{\mathbb{Z}^{d+1}}$ is of finite type. By decomposing $\mathbb{Z}^{d+1}$ into $\mathbb{Z}^d \times \mathbb{Z}$, we may view the system as a one-dimensional shift, that is, a shift of $\mathbb{Z}$, on the alphabet $G^{\mathbb{Z}^d}$. We will then carefully analyze this one-dimensional shift and construct a non-increasing sequence of closed, shift-invariant subgroups of $G^{\mathbb{Z}^d}$. Using our inductive hypothesis, we will find that V is of finite type with respect to this one-dimensional system. That is, we will obtain a finite window $F \subseteq \mathbb{Z}$ such that V is determined by the constraints $\pi_{\mathbb{Z}^d\times F}(V)$. Finally, we will reduce to the d-dimensional shift on $(G^F)^{\mathbb{Z}^d}$. By finiteness of F, G^F is again a compact Lie group and we can apply our inductive hypothesis a second time to obtain another finite window $W \subseteq \mathbb{Z}^d$ such that V is determined by the constraints $\pi_{W\times F}(V)$. This will imply that V is of finite type under the full $(d+1)$-dimensional shift.

Proof of Theorem 2.8. We proceed by induction on the dimension $d \geq 0$.

For the base case $d = 0$, we observe that any closed, shift-invariant subgroup of $G^{\mathbb{Z}^0}$ is of finite type by selecting the finite window $F = \mathbb{Z}^0 = \{0\}$. By Proposition 2.12, we thus conclude that the shift-action $\sigma : \mathbb{Z}^0 \to \mathrm{Aut}(G^{\mathbb{Z}^0})$ satisfies the descending chain condition for every compact Lie group G.

Now, assume the statement holds for an arbitrary but fixed integer $d \geq 0$. Let G be a compact Lie group, and let σ denote the shift action of $\mathbb{Z}^{d+1}$ on $G^{\mathbb{Z}^{d+1}}$. For any subset $S \subseteq \mathbb{Z}^{d+1}$, let $\pi_S : G^{\mathbb{Z}^{d+1}} \to G^S$ denote the canonical projection and observe that π_S is an open, closed, and continuous group homomorphism. Indeed, any canonical projection from a product space is open and continuous, and closedness follows from the fact that $G^{\mathbb{Z}^{d+1}}$ is compact and G^S is Hausdorff.

Let $V \subseteq G^{\mathbb{Z}^{d+1}}$ be a closed, σ-invariant subgroup. Our goal is to show that V is of finite type and thus conclude via Proposition 2.12. We proceed in two steps.

Reduction to the 1-dimensional shift:

We observe that the restriction $\sigma|_{\{0\}^d\times\mathbb{Z}}$ of σ to $\{0\}^d \times \mathbb{Z}$ may be viewed as the shift-action of $\mathbb{Z}$ on the space $(G^{\mathbb{Z}^d})^{\mathbb{Z}}$, i.e., the one-dimensional shift on all bi-infinite words in the alphabet $G^{\mathbb{Z}^d}$. We will now show that, viewed as such, V is of finite type.

For $k \geq 1$, we define the sets $X_k \subseteq G^{\mathbb{Z}^d}$ as

$$X_k = \pi_{\mathbb{Z}^d\times\{k\}}\left(\left\{x \in V : \pi_{\mathbb{Z}^d\times\{0,\ldots,k-1\}}(x) = \left(1_{G^{\mathbb{Z}^d}}, \ldots, 1_{G^{\mathbb{Z}^d}}\right)\right\}\right).$$

Conceptually, X_k is the set of all “followers” in V of the trivial string $\left[1_{G^{\mathbb{Z}^d}}, \ldots, 1_{G^{\mathbb{Z}^d}}\right]$ of length k with respect to the one-dimensional shift. Since the canonical projections are closed and continuous group homomorphisms, the sets X_k are closed subgroups of $G^{\mathbb{Z}^d}$. We further observe that, by construction, the sets X_k are invariant under the shift-action of $\mathbb{Z}^d$ and form a non-increasing sequence. Thus, by our inductive hypothesis, there exists an integer $K \geq 1$ such that $X_K = X_k$ for all $k \geq K$. We set $X = X_K$.

We set $H = \pi_{\mathbb{Z}^d \times \{0,\dots,K\}}(V)$ and

$$V' = \left\{ x \in G^{\mathbb{Z}^{d+1}} : \pi_{\mathbb{Z}^d \times \{0,\dots,K\}} \circ \sigma_{(0,\dots,0,m)}(x) \in H \text{ for all } m \in \mathbb{Z} \right\}.$$

We claim that $V = V'$, which would show that V is of finite type with respect to the one-dimensional shift on the alphabet $G^{\mathbb{Z}^d}$. The inclusion $V \subseteq V'$ is trivial by construction. The reverse inclusion requires more work.

Let $z \in V'$. Since V is a closed subspace of the product space $G^{\mathbb{Z}^{d+1}}$, it is enough to show that for every integer $N \geq K$, there exists an element $y \in V$ such that z and y coincide on the slice $\mathbb{Z}^d \times \{-N, \dots, N\}$.

So, let $N \geq K$. By the definition of V' and the shift-invariance of V, there exists a $y^{(0)} \in V$ such that

$$\pi_{\mathbb{Z}^d \times \{-N,\dots,-N+K\}}(z) = \pi_{\mathbb{Z}^d \times \{-N,\dots,-N+K\}}(y^{(0)}).$$

We assume inductively that there exists a $y^{(j)} \in V$ that coincides with z on the slice $\mathbb{Z}^d \times \{-N, \dots, -N+K+j\}$ for some $j \geq 0$. We observe that by construction the projections of z and $y^{(j)}$ to the slice $S = \mathbb{Z}^d \times \{-N+j+1, \dots, -N+K+j+1\}$ of length $K+1$ both lie in H[2], and they coincide everywhere except possibly on $\mathbb{Z}^d \times \{-N+K+j+1\}$. Hence,

$$\pi_S(y^{(j)})^{-1} \pi_S(z) = \left(1_{G^{\mathbb{Z}^d}}, \dots, 1_{G^{\mathbb{Z}^d}}, x\right)$$

for some $x \in X$. Since $X = X_K = X_{K+j+1}$, we in particular obtain that $x \in X_{K+j+1}$. This implies, by shift-invariance of V, that there exists a $\tilde{x} \in V$ such that

$$\pi_{\mathbb{Z}^d \times \{-N,\dots,-N+K+j+1\}}(\tilde{x}) = \left(1_{G^{\mathbb{Z}^d}}, \dots, 1_{G^{\mathbb{Z}^d}}, x\right).$$

So, by setting $y^{(j+1)} = y^{(j)} \cdot \tilde{x}$, we obtain our desired element in V that coincides with z on the slice $\mathbb{Z}^d \times \{-N, \dots, -N+K+j+1\}$. By induction, this proves that there exists a $y = y^{(2N-K)} \in V$ such that $\pi_{\mathbb{Z}^d \times \{-N,\dots,N\}}(y) = \pi_{\mathbb{Z}^d \times \{-N,\dots,N\}}(z)$, which concludes the proof of the claim that $V = V'$.

Reconstruction via the d-dimensional shift:

Let $\tilde{\sigma}$ denote the shift-action of $\mathbb{Z}^d$ on $(G^{K+1})^{\mathbb{Z}^d}$. Recalling that $H = \pi_{\mathbb{Z}^d \times \{0,\dots,K\}}(V)$, we see that we may view H as a closed and $\tilde{\sigma}$-invariant subgroup of $(G^{K+1})^{\mathbb{Z}^d}$. We observe that the finite product G^{K+1} is again a compact Lie group, so we may apply our inductive assumption and Proposition 2.12 to obtain that H is of finite type with respect to $\tilde{\sigma}$, i.e., there exists a finite subset $W \subseteq \mathbb{Z}^d$ such that

$$H = \left\{ x \in (G^{K+1})^{\mathbb{Z}^d} : \pi_W(\tilde{\sigma}_{\mathbf{m}}(x)) \in \pi_W(H) \text{ for all } \mathbf{m} \in \mathbb{Z}^d \right\}.$$

Using that $V = V'$, we thus obtain

$$V = \left\{ x \in G^{\mathbb{Z}^{d+1}} : \pi_{W \times \{0,\dots,K\}}(\sigma_{\mathbf{n}}(x)) \in \pi_{W \times \{0,\dots,K\}}(V) \text{ for all } \mathbf{n} \in \mathbb{Z}^{d+1} \right\},$$

which shows that V is of finite type and thus concludes the proof. □

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[2] Strictly speaking, this constitutes an abuse of notation. We defined H as a projection to the coordinates $\mathbb{Z}^d \times \{0, \dots, K\}$, whereas the slice here is $S = \mathbb{Z}^d \times \{-N+j+1, \dots, -N+K+j+1\}$. However, since V and V' are invariant under the shift $\sigma_{(0,\dots,0,m)}$ for all $m \in \mathbb{Z}$, we may identify these projected spaces via the shift, making the deduction valid.

2.2 Spectral Characterizations of Ergodicity & Mixing

In the previous section, we used the irreducible unitary representations of X to construct conjugacies to Lie subshifts. In this section, we will continue to rely on these unitary representations, though from a different perspective. We will focus on the Hilbert space $L^2(X, \lambda_X)$ and use the Peter–Weyl theorem to obtain a link to the irreducible unitary representations. By analyzing the interaction between the unitary operators on $L^2(X, \lambda_X)$ induced by the action α and the Peter–Weyl decomposition, we will demonstrate that the probabilistic behavior of the system (ergodicity and mixing) is determined by how the acting group Γ (via these unitary operators) permutes the irreducible unitary representations of X.

Theorem 2.13. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Then the following are equivalent:*

- *(a) α is non-ergodic.*
- *(b) There exists a non-trivial irreducible unitary representation τ of X on a (finite-dimensional) complex Hilbert space $\mathcal{K}$ such that the group*

$$\Gamma_\tau = \{\gamma \in \Gamma : \textit{the representation } \tau \circ \alpha_\gamma \textit{ is unitarily equivalent to } \tau\}$$

has finite index in Γ.

- *(c) There exists a closed, normal, α-invariant subgroup $V \subsetneq X$ and a compatible metric δ on X/V such that X/V is a matrix Lie group, and δ is invariant under $\alpha^{X/V}$.*

In order to illustrate the idea behind the proof, we will consider the special case where X is abelian and $\Gamma = \mathbb{Z}$. That is, we consider the case of a single continuous automorphism $T : X \to X$ on a compact metrizable abelian group X.

Assuming that T is non-ergodic, Proposition 1.18 implies that there exists a non-constant function $f \in L^2(X, \lambda_X)$ such that $f = f \circ T$. Since X is a compact abelian group, the Peter–Weyl theorem (Theorem B.6) implies that the dual group of characters $\widehat{X}$ forms a Hilbert basis of $L^2(X, \lambda_X)$. In other words, $L^2(X, \lambda_X)$ decomposes into a Hilbert direct sum of the one-dimensional subspaces spanned by the characters of X. This allows us to write $f = \sum_{\chi \in \widehat{X}} c_\chi \chi$, for some $c_\chi \in \mathbb{C}$. Since f is non-constant, there exists a non-trivial character $\tau \in \widehat{X}$ such that $c_\tau \neq 0$. By T-invariance, we have

$$\sum_{\chi \in \widehat{X}} c_\chi \chi = f = f \circ T = \sum_{\chi \in \widehat{X}} c_\chi (\chi \circ T),$$

which in particular implies that $c_\tau = c_{\tau \circ T} = c_{\tau \circ T^k}$ for all $k \in \mathbb{Z}$. Again using the fact that the characters form a Hilbert basis, we obtain $\|f\|_2^2 = \sum_{\chi \in \widehat{X}} |c_\chi|^2 < \infty$, which implies that the set $\{\tau \circ T^k : k \in \mathbb{Z}\}$ must be finite. In other words, the subgroup

$$\{k \in \mathbb{Z} : \tau \circ T^k = \tau\}$$

has finite index in $\mathbb{Z}$.

The above proves the implication (a) $\implies$ (b) in the given special case. In the non-abelian case, the characters of X generalize to irreducible unitary representations of X. The Peter–Weyl decomposition will also take a slightly more complicated form, but the proof will essentially be the same. The existence of such a non-constant function will force the "α-orbits" of the non-zero projections of f to the components of the Peter–Weyl decomposition to be finite.

Even though we will not do so for the proof of the general case, let us also discuss how the implication (b) $\implies$ (a) works for our special case $T : X \to X$ as above. So assume that there exists a non-trivial character $\tau \in \widehat{X}$ such that the subgroup $H = \{k \in \mathbb{Z} : \tau = \tau \circ T^k\}$ has finite index in $\mathbb{Z}$. Let $\ell \in \mathbb{Z}_{\geq 1}$ be the index $[\mathbb{Z} : H]$. Then $\{0, \dots, \ell - 1\}$ forms a set of representatives of $\mathbb{Z}/H$ and we see that the function

$$f = \sum_{k=0}^{\ell-1} \tau \circ T^k \in L^2(X, \lambda_X)$$

is non-constant and invariant under T.

We note in passing that the equivalence of (a) and (b) in the special case we just proved can be restated in terms of the associated *Koopman operator* $U : L^2(X, \lambda_X) \to L^2(X, \lambda_X)$, defined by $f \mapsto f \circ T$. Specifically, the automorphism T is ergodic if and only if U has a purely continuous spectrum on the orthogonal complement of the constant functions in $L^2(X, \lambda_X)$ (or equivalently, if U has no non-constant eigenfunctions). Hence, the ergodicity of the system is completely characterized by the spectrum of U. Furthermore, in this special case, ergodicity and mixing coincide. Thus, mixing is also entirely determined by the spectrum of the Koopman operator.

Next, let us see how (b) $\implies$ (c) works in our special case. Let $\tau \in \widehat{X}$ and $\ell \in \mathbb{Z}_{\geq 1}$ be as in the paragraph above. For each $x \in X$, we set $\eta(x)$ to be the diagonal unitary matrix

$$\eta(x) = \begin{pmatrix} \tau(x) & & & \\ & \tau \circ T(x) & & \\ & & \ddots & \\ & & & \tau \circ T^{\ell-1}(x) \end{pmatrix}.$$

Then η defines a continuous group homomorphism from X into the group of unitary matrices $\mathrm{U}(\ell)$. We observe that

$$\eta \circ T(x) = \begin{pmatrix} \tau \circ T(x) & & & \\ & \tau \circ T^2(x) & & \\ & & \ddots & \\ & & & \tau(x) \end{pmatrix},$$

where we used the fact that $\tau \circ T^\ell = \tau$. So there exists a permutation matrix P such that

$$\eta \circ T^k(x) = P^k \eta(x) P^{-k}, \tag{2.3}$$

for all $x \in X$. We set $V = \ker \eta$ and observe that V is a closed, normal subgroup. Furthermore, by Eq. (2.3) we see that V is invariant under T and the fact that τ was non-trivial yields that V is a proper subgroup. Then, X/V is topologically isomorphic (that is, there is a continuous group isomorphism with continuous inverse) to the matrix Lie group $\eta(X)$ and T on X/V corresponds to conjugation by permutation matrices on $\eta(X)$, showing that it acts isometrically.

We omit discussing the implication (c) $\implies$ (b) for the special case as the general case does not present any additional difficulties.

Proof of Theorem 2.13. (a) $\implies$ (b): We set

$$\mathcal{H} := L^2_0(X, \lambda_X) = \left\{ f \in L^2(X, \lambda_X) : \int_X f \, d\lambda_X = 0 \right\}$$

and consider the right regular representation of X on $\mathcal{H}$ given by $(\rho(x)f)(y) = f(yx)$ for $f \in \mathcal{H}$ and $x, y \in X$. Since α is non-ergodic, Proposition 1.18 gives a non-zero α-invariant function $f \in \mathcal{H}$. By the Peter–Weyl theorem (Theorem B.7), the right regular representation $(\rho, \mathcal{H})$ decomposes as a direct sum of non-trivial irreducible unitary representations of X, which are necessarily finite-dimensional. So we may find such a subrepresentation $(\tau, \mathcal{K})$ of $(\rho, \mathcal{H})$ such that the orthogonal projection $P_{\mathcal{K}} f$ of f onto $\mathcal{K}$ is non-zero.

We fix a $\gamma \in \Gamma$ for now and define the operator W_γ on $\mathcal{H}$ by $W_\gamma h = h \circ \alpha_\gamma$. Since α_γ^{-1} preserves the Haar measure, W_γ is unitary. Since f is α-invariant, we have $W_\gamma^{-1} f = f$. Combining these observations, we obtain

$$\|P_{\mathcal{K}} f\| = \|P_{\mathcal{K}} W_\gamma^{-1} f\| = \|W_\gamma P_{\mathcal{K}} W_\gamma^{-1} f\| = \|P_{W_\gamma \mathcal{K}} f\|, \tag{2.4}$$

where $P_{W_\gamma \mathcal{K}}$ is the orthogonal projection onto the subspace $W_\gamma \mathcal{K} \subseteq \mathcal{H}$. We consider the unitary representation τ_γ of X on $\mathcal{K}$ defined by $\tau_\gamma(x) = \tau \circ \alpha_\gamma(x)$. For $x \in X$ and $h \in \mathcal{K}$ we compute

$$\begin{aligned}
(W_\gamma^{-1} \rho(x) W_\gamma(h))(y) &= (\rho(x) W_\gamma(h))(\alpha_\gamma^{-1}(y)) \\
&= (W_\gamma(h))(\alpha_\gamma^{-1}(y)x) \\
&= h(y \alpha_\gamma(x)) \\
&= (\rho(\alpha_\gamma(x))(h))(y) \\
&= (\tau_\gamma(x)(h))(y),
\end{aligned}$$

for every $y \in X$. This shows that τ_γ is unitarily equivalent to ρ restricted to $W_\gamma \mathcal{K}$.

We notice that for $\gamma_1, \gamma_2 \in \Gamma$, we have that τ_{γ_1} is unitarily equivalent to τ_{γ_2} if and only if there exists a unitary operator U of $\mathcal{K}$ such that $\tau(\alpha_{\gamma_1}(x)) = U^{-1} \tau(\alpha_{\gamma_2}(x)) U$ for all $x \in X$. By replacing x with $\alpha_{\gamma_2}^{-1}(x)$, we see that the latter is in turn equivalent to $\gamma_1 \gamma_2^{-1} \in \Gamma_\tau$. In particular, if $\Gamma_\tau \gamma_1 \neq \Gamma_\tau \gamma_2$, then τ_{γ_1} and τ_{γ_2} are not unitarily equivalent.

Finally, let $\gamma \in \Gamma$ and assume that τ_γ is not unitarily equivalent to τ. Then, since τ_γ is unitarily equivalent to ρ restricted to $W_\gamma \mathcal{K}$, we obtain that $\mathcal{K}$ is orthogonal to $W_\gamma \mathcal{K}$. Combining this with Eq. (2.4), we see that this can happen for at most finitely many distinct $W_\gamma \mathcal{K}$. We thus obtain that Γ_τ has finite index in Γ.

(b) $\Longrightarrow$ (c): Let $(\tau, \mathcal{K})$ be the given representation of X. As we have seen above, $\tau \circ \alpha_{\gamma_1}$ and $\tau \circ \alpha_{\gamma_2}$ are unitarily equivalent if and only if $\Gamma_\tau \gamma_1 = \Gamma_\tau \gamma_2$, for $\gamma_1, \gamma_2 \in \Gamma$. We fix representatives $\gamma_1, \ldots, \gamma_m$ of the equivalence classes in $\Gamma_\tau \backslash \Gamma$ and define the non-trivial unitary representation $\eta = \oplus_{i=1}^m \tau_{\gamma_i}$ of X on the finite-dimensional Hilbert space $\mathcal{K}^{\oplus m}$, where $\tau_\gamma(x) = \tau \circ \alpha_\gamma(x)$ for $x \in X$ and $\gamma \in \Gamma$, as above.

We fix some $\gamma \in \Gamma$ and notice that right translation by γ defines a bijection on $\{\Gamma_\tau \gamma_1, \ldots, \Gamma_\tau \gamma_m\}$, say $\Gamma_\tau \gamma_{\phi(i)} = \Gamma_\tau \gamma_i \gamma$ for some permutation $\phi : \{1, \ldots, m\} \to \{1, \ldots, m\}$. Then, by the above, there exist unitary operators of $\mathcal{K}$ that intertwine $\tau_{\gamma_i \gamma}$ and $\tau_{\gamma_{\phi(i)}}$ for all i. Combining these unitary operators with a permutation of the m components, we obtain a unitary operator U_γ of $\mathcal{K}^{\oplus m}$ such that

$$\eta \circ \alpha_\gamma(x) = \bigoplus_{i=1}^m \tau_{\gamma_i} \circ \alpha_\gamma = \bigoplus_{i=1}^m \tau_{\gamma_i \gamma} = U_\gamma \eta(x) U_\gamma^{-1}. \tag{2.5}$$

We now set $Y = \eta(X)$. By properties of η, Y is a compact non-trivial subgroup of the group of unitary operators $\mathcal{U}(\mathcal{K}^{\oplus m})$ on the finite-dimensional complex Hilbert space $\mathcal{K}^{\oplus m}$. As such, it is a compact matrix Lie group. We define an action β of Γ on Y by $\beta_\gamma(y) = U_\gamma y U_\gamma^{-1}$ for all $y \in Y$ and $\gamma \in \Gamma$. Note that by Eq. (2.5), we have $\eta \circ \alpha_\gamma = \beta_\gamma \circ \eta$

for all $\gamma \in \Gamma$. Since the operator norm is invariant under unitary operators, β preserves the metric $\tilde{\delta}$ induced by the operator norm.

Let $V = \ker(\eta)$. Since η is non-trivial and Y is Hausdorff, V is a proper, closed, normal subgroup of X. Using Eq. (2.5), we further obtain that V is α-invariant. Finally, the above gives that $X/V \cong Y$ is a matrix Lie group and since $\eta \circ \alpha_\gamma^{X/V} = \beta_\gamma \circ \eta$ for all $\gamma \in \Gamma$, we see that $\alpha^{X/V}$ preserves the metric δ on X/V corresponding to the metric $\tilde{\delta}$ on Y.

(c) $\Longrightarrow$ (a): Let $V \subsetneq X$ be as described in the statement of the theorem. Since X/V is non-trivial, we may find a closed ball B around the identity such that B has non-empty interior and $B \subsetneq X/V$. As $\alpha^{X/V}$ preserves the metric δ on X/V, B is $\alpha^{X/V}$-invariant. Thus, the pre-image of B under the quotient map $X \to X/V$ is a closed, α-invariant, proper subset with non-empty interior, which implies that α is non-ergodic. ☐

Theorem 2.14. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Then the following are equivalent:*

(a) α is mixing.
(b) For every infinite subgroup $\Delta \subseteq \Gamma$, the restriction of α to Δ is mixing on X.
(c) For every infinite subgroup $\Delta \subseteq \Gamma$, the restriction of α to Δ is ergodic on X.
(d) The group

$$\Gamma_\tau = \{\gamma \in \Gamma : \textit{the representation } \tau \circ \alpha_\gamma \textit{ is unitarily equivalent to } \tau\}$$

is finite for every non-trivial irreducible unitary representation τ of X.

Proof. (a) $\Longrightarrow$ (b): This follows immediately from the definition.

(b) $\Longrightarrow$ (c): See Lemma 1.20.

(c) $\Longrightarrow$ (d): By contraposition, assume there exists a non-trivial, continuous, irreducible, unitary representation τ of X such that Γ_τ is infinite. Then Theorem 2.13 implies that the restriction of α to the infinite group Γ_τ is non-ergodic.

(d) $\Longrightarrow$ (a): As in the proof of Theorem 2.13, let $\mathcal{H} = \{f \in L^2(X, \lambda_X) : \int_X f \, d\lambda_X = 0\}$ and denote by ρ the right regular representation of X on $\mathcal{H}$. Assume τ is the restriction of ρ to a finite-dimensional, ρ-invariant subspace $\mathcal{K}$ of $\mathcal{H}$. We define $\mathcal{K}_\gamma = \{f \circ \alpha_\gamma : f \in \mathcal{K}\}$ for $\gamma \in \Gamma$ and claim that the set $\Gamma(\mathcal{K}) = \{\gamma \in \Gamma : \mathcal{K}_\gamma \text{ is not orthogonal to } \mathcal{K}\}$ is finite.

We observe that $(\tau, \mathcal{K})$ decomposes into a direct sum $\bigoplus_{i=1}^m (\tau_i, \mathcal{K}_i)$ of irreducible unitary representations (by Corollary B.8). Let $\gamma \in \Gamma$. In the proof of Theorem 2.13, we have seen that $(\rho|_{\mathcal{K}_\gamma}, \mathcal{K}_\gamma)$ is unitarily equivalent to $(\tau \circ \alpha_\gamma, \mathcal{K}) = \bigoplus_{i=1}^m (\tau_i \circ \alpha_\gamma, \mathcal{K}_i)$. Thus, if $\mathcal{K}_\gamma$ is not orthogonal to $\mathcal{K}$, then there exist $i, j \in \{1, \ldots, m\}$ such that $\tau_i \circ \alpha_\gamma \cong \tau_j$. So to show that $\Gamma(\mathcal{K})$ is finite, it is enough to show that for fixed $i, j \in \{1, \ldots, m\}$, there exist only finitely many $\gamma \in \Gamma$ with $\tau_i \circ \alpha_\gamma \cong \tau_j$. And indeed, if i, j are fixed and we assume that there is at least one $\gamma_0 \in \Gamma$ such that $\tau_i \circ \alpha_{\gamma_0} \cong \tau_j$, then for any $\gamma \in \Gamma$, $\tau_i \circ \alpha_\gamma \cong \tau_j$ holds if and only if $\tau_i \circ \alpha_{\gamma_0} \cong \tau_i \circ \alpha_\gamma$. And the latter is the case if and only if $\gamma_0 \gamma^{-1} \in \Gamma_{\tau_i}$, i.e., if they belong to the same coset of the finite subgroup Γ_{τ_i}. This proves our claim.

We are now ready to prove that α is mixing. By part (c) of Proposition 1.21, it is enough to show that $\lim_{\gamma \to \infty} \langle f, g \circ \alpha_\gamma \rangle = 0$ for all $f, g \in \mathcal{H}$. Let $f, g \in \mathcal{H}$, let $(\gamma_n) \subseteq \Gamma$ be a sequence going to infinity as $n \to \infty$, and let $\varepsilon > 0$. By the Peter–Weyl theorem (Theorem B.7), $\mathcal{H}$ decomposes into a direct sum of irreducible, finite-dimensional subrepresentations. Thus, there exists a finite-dimensional, ρ-invariant subspace $\mathcal{K}$ of $\mathcal{H}$ such that $\|P_{\mathcal{K}} f - f\| + \|P_{\mathcal{K}} g -$

$g\| < \varepsilon/(\|f\| + \|g\| + 1)$, where $P_{\mathcal{K}}$ denotes the orthogonal projection onto $\mathcal{K}$. We compute

$$\begin{aligned} |\langle P_{\mathcal{K}} f, P_{\mathcal{K}} g \circ \alpha_\gamma \rangle - \langle f, g \circ \alpha_\gamma \rangle| &= |\langle P_{\mathcal{K}} f, (P_{\mathcal{K}} g - g) \circ \alpha_\gamma \rangle + \langle P_{\mathcal{K}} f - f, g \circ \alpha_\gamma \rangle| \\ &\leq \|P_{\mathcal{K}} f\| \|P_{\mathcal{K}} g - g\| + \|P_{\mathcal{K}} f - f\| \|g\| \\ &< \varepsilon, \end{aligned}$$

for any $\gamma \in \Gamma$. Now using the fact that $\Gamma(\mathcal{K})$ is finite and that the sequence (γ_n) goes to infinity, there exists an integer $N \geq 1$ such that $\gamma_n \notin \Gamma(\mathcal{K})$ for all $n \geq N$. Consequently, for all $n \geq N$, the spaces $\mathcal{K}$ and $\mathcal{K}_{\gamma_n}$ are orthogonal, meaning $\langle P_{\mathcal{K}} f, P_{\mathcal{K}} g \circ \alpha_{\gamma_n} \rangle = 0$. It follows that $\limsup_{n\to\infty} |\langle f, g \circ \alpha_{\gamma_n} \rangle| \leq \varepsilon$, and since $\varepsilon > 0$ was arbitrary, this concludes the proof. □

As an application of Theorem 2.13, we will prove an analogue of Lemma 1.29.

Proposition 2.15. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action and $Y \subseteq X$ a closed, normal, and α-invariant subgroup. If the induced actions α^Y and $\alpha^{X/Y}$ are ergodic (respectively, mixing), then α is ergodic (respectively, mixing).*

Proof. By contraposition, assume that α is not ergodic. Then, by Theorem 2.13, there exists a non-trivial irreducible unitary representation τ of X such that the stabilizer $\Gamma_\tau = \{\gamma \in \Gamma : \tau \circ \alpha_\gamma \simeq \tau\}$ has finite index in Γ. We distinguish two cases.

If $Y \subseteq \ker \tau$, then τ factors through X/Y, i.e., there exists a non-trivial irreducible unitary representation $\overline{\tau}$ of X/Y such that $\tau = \overline{\tau} \circ \pi$, where $\pi : X \to X/Y$ is the canonical projection. Given $\gamma \in \Gamma$, we observe that $\overline{\tau} \circ \alpha_\gamma^{X/Y} \simeq \overline{\tau}$ if and only if $\overline{\tau} \circ \pi \circ \alpha_\gamma \simeq \overline{\tau} \circ \pi$, which in turn is equivalent to $\tau \circ \alpha_\gamma \simeq \tau$. This shows that $\Gamma_\tau = \Gamma_{\overline{\tau}}$ and thus establishes non-ergodicity of $\alpha^{X/Y}$ by a second application of Theorem 2.13.

Otherwise, we have that $Y \not\subseteq \ker \tau$. Notice that the restriction $\tau|_Y$ of τ to Y defines a unitary representation of Y on a finite-dimensional Hilbert space. By the Peter–Weyl theorem (specifically Corollary B.8), this implies that $\tau|_Y$ decomposes into finitely many irreducible unitary representations of Y, say $\sigma_1, \ldots, \sigma_m$. Since $Y \not\subseteq \ker \tau$, at least one of these components must be non-trivial. We can therefore assume without loss of generality that σ_1 is a non-trivial irreducible unitary representation of Y. For $\gamma \in \Gamma_\tau$, we have by definition that $\tau|_Y \circ \alpha_\gamma^Y \simeq \tau|_Y$. Thus, there is an action of Γ_τ on the finite set S of equivalence classes of the irreducible components $\sigma_1, \ldots, \sigma_m$ of $\tau|_Y$. Since S is finite, any orbit has to be finite and thus the orbit-stabilizer theorem implies that the stabilizer $\{\gamma \in \Gamma_\tau : \sigma_1 \circ \alpha_\gamma^Y \simeq \sigma_1\}$ has finite index in Γ_τ. Combining this with our assumption that Γ_τ has finite index in Γ, we obtain that Γ_{σ_1} has finite index in Γ, implying that α^Y is non-ergodic.

In total, we have shown that if α is non-ergodic, then either $\alpha^{X/Y}$ or α^Y is non-ergodic. The mixing case follows an analogous argument using the characterization in Theorem 2.14 (replacing *finite index* subgroups with *infinite* subgroups). □

Remarkably, in the case of quasi-algebraic actions, the probabilistic notion of ergodicity and the topological notion of transitivity coincide, as the next corollary establishes. Thus, we will not distinguish between these two properties for quasi-algebraic actions and usually only refer to ergodicity from this point on.

Corollary 2.16. *Let $\alpha : \Gamma \to \mathrm{Aut}(X)$ be a quasi-algebraic action. Then α is ergodic if and only if it is topologically transitive.*

Proof. Assume first that α is ergodic. Let $U \subseteq X$ be a non-empty, open, α-invariant subset. Since U is non-empty and open, it must have positive Haar measure (see Theorem A.3).

Thus, ergodicity yields $\lambda_X(U) = 1$ and thus $\overline{U} = X$. By Proposition 1.8 this implies that α is topologically transitive.

Conversely, assume that α is non-ergodic. Then Theorem 2.13 gives a closed, normal, α-invariant subgroup $V \subsetneq X$ and a compatible metric δ on X/V invariant under $\alpha^{X/V}$. Let $B \subseteq X/V$ be an open ball around the identity small enough that it is not dense in X/V. Since $\alpha^{X/V}$ preserves the metric, B is $\alpha^{X/V}$-invariant. We conclude that the pre-image of B under the quotient map $X \to X/V$ is a non-empty, open, α-invariant subset, which is not dense in X. This proves that α is not topologically transitive (again by Proposition 1.8). ☐

We have already mentioned that in the case of a quasi-algebraic action of $\mathbb{Z}$, ergodicity and mixing are equivalent. In fact, a more general statement is true.

Corollary 2.17. *Let* $\alpha : \Gamma \to \mathrm{Aut}(X)$ *be a quasi-algebraic action. Assume that* Γ *is infinite and that every infinite subgroup of* Γ *has finite index. Then* α *is ergodic if and only if it is mixing.*

Proof. This is immediate by combining Theorem 2.13 part (b) with Theorem 2.14 part (d). ☐

Examples of infinite groups in which every infinite subgroup has finite index include $\mathbb{Z}$, $\mathbb{Z} \times F$ for any finite group F, the Prüfer p-group $\mathbb{Z}(p^\infty)$, and any Tarski monster group.

While Theorem 2.13 and Theorem 2.14 could already be applied to our four running examples to study their probabilistic behavior, we postpone this analysis to Chapter 3, where we will develop a more unified algebraic framework. Nevertheless, we invite the reader to think about how these theorems may be applied to our examples.

Chapter 3

The Algebraic Dictionary

In the preceding chapters, we extensively analyzed the behavior of four different dynamical systems: Arnold's Cat Map, the Toral Rotation, the Times Two Map, and the Square Shift. For every dynamical property introduced, we determined whether each system satisfied it. To do so, however, we often employed arguments of different natures, both within the same example for different properties and across different examples. For instance, to show that Arnold's Cat Map is topologically transitive (Example 1.9), we relied on a geometric argument involving stable and unstable manifolds. To show that the Square Shift is topologically transitive (Example 1.14), we provided an explicit constructive argument by "gluing" finite admissible patterns. Finally, to show that Arnold's Cat Map is mixing, we used Fourier analysis on the 2-torus $\mathbb{T}^2$ (Example 1.22).

However, having gone through these proofs using a variety of arguments, we observed an interesting commonality among the examples of toral automorphisms. For a toral automorphism $T_B : \mathbb{T}^d \to \mathbb{T}^d$ induced by a matrix $B \in \mathrm{GL}_d(\mathbb{Z})$, the dynamical behavior of the system was entirely characterized by the spectrum of the defining matrix B (Remarks 1.10, 1.23 and 1.31). Specifically, the presence of eigenvalues on the unit circle or eigenvalues that are roots of unity dictated properties like expansiveness and mixing.

This observation raises a natural question: Is there a generalized "spectrum" for the Square Shift, or for general shift-actions, or even for all quasi-algebraic actions, that similarly characterizes the dynamical behavior of these systems?

The goal of this chapter is to introduce a unifying framework that allows us to answer this affirmatively and thus to analyze the dynamical properties of our examples using a single algebraic approach. At its core lies Pontryagin duality, which bridges the worlds of dynamical systems and commutative algebra. However, because this duality applies specifically to abelian groups, the general class of quasi-algebraic actions is too broad for this approach. We therefore restrict our attention to systems where the group being acted upon is abelian and the acting group is $\mathbb{Z}^d$, for some integer $d \geq 1$. The latter restriction ensures that we obtain algebraic structures that are well-understood in commutative algebra. We formally define this class of systems as follows:

Definition 3.1. For $d \geq 1$, an *algebraic $\mathbb{Z}^d$-action* is an action α of $\mathbb{Z}^d$ on a compact metrizable abelian group X by continuous automorphisms of X. If the dimension d is clear from context, we will often denote the system by (X, α).

To develop this algebraic dictionary, the remainder of this chapter is structured as follows. In Section 3.1, we first establish the fundamental correspondence between algebraic $\mathbb{Z}^d$-actions and countable modules over the ring of Laurent polynomials with integer coefficients in d commuting variables. Next, in Section 3.2, we make the notion of the "spectrum" of an algebraic $\mathbb{Z}^d$-action concrete. We will achieve this by considering certain varieties defined by the associated prime ideals of a given module. Having built the bridge, we will, in Section 3.3, translate dynamical properties of an algebraic $\mathbb{Z}^d$-action into the algebraic landscape as conditions on the “spectrum”. Finally, in Section 3.4, we will discuss many examples, including a final analysis of our four running examples.

The content of this chapter is based on Sections 5 and 6 of Klaus Schmidt’s monograph *Dynamical Systems of Algebraic Origin* [Sch95].

3.1 From Dynamics to Algebra: The Dual Module

Let $d \geq 1$. We will denote by $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integer coefficients in d commuting variables $u_1, \ldots, u_d$. The following proposition establishes the fundamental correspondence between the worlds of dynamical systems and commutative algebra upon which the theory of this chapter relies.

Proposition 3.2. *There is a one-to-one correspondence, up to conjugacy and isomorphism, between algebraic $\mathbb{Z}^d$-actions and countable R_d-modules:*[1]

(a) Given an algebraic $\mathbb{Z}^d$-action (X, α), the Pontryagin dual $\widehat{X}$ is a countable R_d-module, where scalar multiplication is defined by extending the dual automorphisms linearly:

$$f \cdot \chi = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \hat{\alpha}_{\mathbf{n}}(\chi),$$

for any Laurent polynomial $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in R_d$ and any character $\chi \in \widehat{X}$. Conversely, given a countable R_d-module M, multiplication by a monomial $u^{\mathbf{n}} \in R_d$ on M defines a continuous automorphism of the discrete group M for every $\mathbf{n} \in \mathbb{Z}^d$. By Pontryagin duality, this induces an algebraic $\mathbb{Z}^d$-action α^M on the compact metrizable abelian group $X^M = \widehat{M}$.

(b) Let (X, α) and (Y, β) be algebraic $\mathbb{Z}^d$-actions. A continuous group homomorphism $\varphi : X \to Y$ is equivariant, meaning $\varphi \circ \alpha_{\mathbf{n}} = \beta_{\mathbf{n}} \circ \varphi$ for all $\mathbf{n} \in \mathbb{Z}^d$, if and only if the dual homomorphism $\hat{\varphi} : \widehat{Y} \to \widehat{X}$ is an R_d-module homomorphism. Furthermore, φ is surjective (respectively, injective) if and only if $\hat{\varphi}$ is injective (respectively, surjective). In particular, φ is a conjugacy if and only if $\hat{\varphi}$ is an R_d-module isomorphism.

Proof. The proof is essentially an application of the results presented in Section A.3. We first establish the claims made in parts (a) and (b), and subsequently demonstrate that the constructions described in part (a) are mutually inverse up to conjugacy and isomorphism, thereby establishing the one-to-one correspondence.

Let (X, α) be an algebraic $\mathbb{Z}^d$-action. By Propositions A.5 and A.9, we obtain that $\widehat{X}$ is a countable discrete abelian group. For each $\mathbf{n} \in \mathbb{Z}^d$, Proposition A.11 implies that the dual $\hat{\alpha}_{\mathbf{n}}$ of $\alpha_{\mathbf{n}}$ is a continuous automorphism of $\widehat{X}$. By the definition of dual homomorphisms, the identity $\alpha_{\mathbf{n}} \circ \alpha_{\mathbf{m}} = \alpha_{\mathbf{n}+\mathbf{m}}$ is equivalent to $\hat{\alpha}_{\mathbf{m}} \circ \hat{\alpha}_{\mathbf{n}} = \hat{\alpha}_{\mathbf{n}+\mathbf{m}}$. With this, it is readily checked that the scalar multiplication as defined in the statement of the proposition turns $\widehat{X}$ into an R_d-module.

Next, let M be a countable R_d-module. We equip M with the discrete topology. Since the monomial $u^{\mathbf{n}}$ is a unit in R_d, this makes multiplication by $u^{\mathbf{n}}$ into a continuous automorphism

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[1] The commutative diagram featured on the title page of this thesis provides a visual summary of the correspondence established in Proposition 3.2. In category theoretical terms, there is an anti-equivalence between the category of algebraic $\mathbb{Z}^d$-actions and the category of countable R_d-modules.

of M for every $\mathbf{n} \in \mathbb{Z}^d$. Again applying Propositions A.5, A.9 and A.11, we obtain that $X^M = \widehat{M}$ is a compact metrizable abelian group and that the homomorphism dual to multiplication by the monomial $u^{\mathbf{n}}$ is a continuous automorphism of X^M, which we will denote by $\alpha_{\mathbf{n}}^M$. In this case, we observe that the identity $u^{\mathbf{n}} \cdot (u^{\mathbf{m}} \cdot m) = u^{\mathbf{n}+\mathbf{m}} \cdot m$ for every $m \in M$ translates to $\alpha_{\mathbf{m}}^M \circ \alpha_{\mathbf{n}}^M = \alpha_{\mathbf{n}+\mathbf{m}}^M$ for every $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$. This implies that $\alpha^M : \mathbb{Z}^d \to \mathrm{Aut}(X^M)$ given by $\mathbf{n} \mapsto \alpha_{\mathbf{n}}^M$ defines an algebraic $\mathbb{Z}^d$-action.

Now, let (X, α) and (Y, β) be algebraic $\mathbb{Z}^d$-actions, and let $\varphi : X \to Y$ be a continuous group homomorphism. Then, the identity $\varphi \circ \alpha_{\mathbf{n}} = \beta_{\mathbf{n}} \circ \varphi$ is equivalent to $\hat{\alpha}_{\mathbf{n}} \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\beta}_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. By construction, $\hat{\alpha}_{\mathbf{n}}$ and $\hat{\beta}_{\mathbf{n}}$ correspond to multiplication by the monomial $u^{\mathbf{n}}$. We thus see that φ is equivariant if and only if $\hat{\varphi}$ is R_d-linear. Furthermore, Proposition A.11 implies that φ is surjective if and only if $\hat{\varphi}$ is injective and that φ is injective if and only if $\hat{\varphi}$ is surjective. Since X and Y are compact Hausdorff spaces, any continuous bijection $\varphi : X \to Y$ is necessarily a homeomorphism. Thus, φ is a conjugacy if and only if $\hat{\varphi}$ is an R_d-module isomorphism.

It remains to show that the constructions of part (a) are mutually inverse up to conjugacy and isomorphism.

Let (X, α) be an algebraic $\mathbb{Z}^d$-action. By Pontryagin duality (Theorem A.6), we may canonically identify X with its double dual (i.e., the dual group of $\widehat{X}$) via the pairing $\langle x, \chi \rangle = \chi(x)$ for $x \in X$ and $\chi \in \widehat{X}$. The identity $\langle \alpha_{\mathbf{n}}(x), \chi \rangle = \chi(\alpha_{\mathbf{n}}(x)) = \langle x, \hat{\alpha}_{\mathbf{n}}(\chi) \rangle$ for all $\mathbf{n} \in \mathbb{Z}^d$ then implies that (X, α) is conjugate to $(X^{\widehat{X}}, \alpha^{\widehat{X}})$. This shows that, up to conjugacy, every algebraic $\mathbb{Z}^d$-action arises from a countable R_d-module via the construction of part (a). Conversely, an analogous argument shows that, up to isomorphism, every countable R_d-module arises from an algebraic $\mathbb{Z}^d$-action via the construction of part (a).

Finally, assume that two algebraic $\mathbb{Z}^d$-actions (X, α) and (Y, β) give rise to isomorphic R_d-modules $\widehat{X}$ and $\widehat{Y}$. Let $\theta : \widehat{Y} \to \widehat{X}$ be an isomorphism. Then, by identifying the duals of $\widehat{X}$ and $\widehat{Y}$ with X and Y, respectively, and using part (b), we obtain a conjugacy $\hat{\theta} : X \to Y$. This shows that an algebraic $\mathbb{Z}^d$-action from which a given countable R_d-module arises is unique up to conjugacy. Conversely, an analogous argument shows that a countable R_d-module from which a given algebraic $\mathbb{Z}^d$-action arises is unique up to isomorphism.

In total, we have shown that there is a one-to-one correspondence, up to conjugacy and isomorphism, between algebraic $\mathbb{Z}^d$-actions and countable R_d-modules. □

There is also a correspondence between their subobjects.

Proposition 3.3. *Let (X, α) be an algebraic $\mathbb{Z}^d$-action and let $M = \widehat{X}$ be the corresponding R_d-module. Then there is an inclusion-reversing bijection between the submodules of M and the closed, α-invariant subgroups of X.*

Proof. By Pontryagin duality, taking the annihilator defines an inclusion-reversing bijection between the closed subgroups of X and the subgroups of its discrete dual M (see Proposition A.7). It thus remains to show that a closed subgroup $Y \subseteq X$ is α-invariant if and only if its annihilator $Y^\perp$ is an R_d-submodule of M.

Since $Y = (Y^\perp)^\perp$, we have that Y is α-invariant if and only if every character of $Y^\perp$ annihilates $\alpha_{\mathbf{n}}(Y)$ for all $\mathbf{n} \in \mathbb{Z}^d$, which can be written as

$$\langle \alpha_{\mathbf{n}}(y), \chi \rangle = 1,$$

for all $y \in Y$, $\chi \in Y^\perp$, and $\mathbf{n} \in \mathbb{Z}^d$.[2] By the definition of the R_d-module structure on M, we

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[2] If A is a locally compact abelian group, $x \in A$ and $\chi \in \widehat{A}$, then we denote the evaluation of χ at x by $\langle x, \chi \rangle$, i.e., $\langle x, \chi \rangle = \chi(x)$. This notation is standard and is due to the symmetric nature of the spaces A and $\widehat{A}$ (any locally compact abelian group is canonically isomorphic to its double dual by Pontryagin duality).

have $\langle \alpha_{\mathbf{n}}(y), \chi \rangle = \langle y, u^{\mathbf{n}} \cdot \chi \rangle$. Therefore, the previous condition is equivalent to

$$\langle y, u^{\mathbf{n}} \cdot \chi \rangle = 1$$

for all $y \in Y$, $\chi \in Y^{\perp}$, and $\mathbf{n} \in \mathbb{Z}^d$. That is, $u^{\mathbf{n}} \cdot \chi \in Y^{\perp}$ for all $\mathbf{n} \in \mathbb{Z}^d$ and $\chi \in Y^{\perp}$, which is precisely the condition for $Y^{\perp}$ to be an R_d-submodule. $\square$

Example 3.4. Let $M = R_d$. We notice that, as an abelian group, R_d is isomorphic to the direct sum $\bigoplus_{\mathbf{n} \in \mathbb{Z}^d} \mathbb{Z}$ of copies of $\mathbb{Z}$ indexed by $\mathbb{Z}^d$. This can be seen, for example, by considering the canonical $\mathbb{Z}$-module basis of R_d given by the monomials. By Propositions A.8 and A.10, we thus obtain

$$X^{R_d} = \widehat{R_d} \cong \prod_{\mathbf{n} \in \mathbb{Z}^d} \mathbb{T} = \mathbb{T}^{\mathbb{Z}^d},$$

where we use the fact that $\widehat{\mathbb{Z}} = \mathbb{T}$. We make the following identification of X^{R_d} with $\mathbb{T}^{\mathbb{Z}^d}$. For every $x = (x_{\mathbf{n}})$ in $\mathbb{T}^{\mathbb{Z}^d}$, we set

$$\langle x, f \rangle = \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}} \right),$$

for every $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}}$ in R_d. This defines an isomorphism $\mathbb{T}^{\mathbb{Z}^d} \ni x \mapsto \langle x, \cdot \rangle \in X^{R_d}$. Under this identification we have

$$\begin{aligned}
\langle \alpha_{\mathbf{m}}^{R_d}(x), f \rangle &= \langle x, \hat{\alpha}_{\mathbf{m}}^{R_d}(f) \rangle \\
&= \langle x, u^{\mathbf{m}} \cdot f \rangle \\
&= \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{u^{\mathbf{m}} \cdot f}(\mathbf{n}) x_{\mathbf{n}} \right) \\
&= \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n} - \mathbf{m}) x_{\mathbf{n}} \right) \\
&= \langle \sigma_{\mathbf{m}}(x), f \rangle,
\end{aligned}$$

for all $x \in \mathbb{T}^{\mathbb{Z}^d}$, $f \in R_d$ and $\mathbf{m} \in \mathbb{Z}^d$. We thus see that α^{R_d} coincides with the shift-action on $\mathbb{T}^{\mathbb{Z}^d}$.

By a straightforward generalization, the same reasoning shows that the algebraic $\mathbb{Z}^d$-action dual to the free module R_d^n is conjugate to the shift-action of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$.

To further illustrate the correspondence established in Proposition 3.2, we will determine the dual module of two of our running examples.

Example 3.5 (Arnold's Cat Map V)**.** Recall that Arnold's Cat Map is defined as the toral automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Crucially, we note that $\mathbb{T}^2$ is abelian and that we have already determined T_A to be a quasi-algebraic action of $\mathbb{Z}$. Thus, we see that $(\mathbb{T}^2, T_A)$ is an algebraic $\mathbb{Z}$-action. Using Propositions A.8 and A.10, we obtain $\widehat{\mathbb{T}^2} = \mathbb{Z}^2$. To see the R_1-module structure of $\mathbb{Z}^2$, we must determine $\hat{T}_A$. Let $x \in \mathbb{T}^2$ and $\mathbf{v} \in \mathbb{Z}^2$. We compute

$$\begin{aligned}
\left\langle x, \hat{T}_A(\mathbf{v}) \right\rangle &= \langle T_A(x), \mathbf{v} \rangle \\
&= \exp\left(2\pi i \left(A\tilde{x} \cdot \mathbf{v}\right)\right) \\
&= \exp\left(2\pi i \left(\tilde{x} \cdot A^T \mathbf{v}\right)\right) \\
&= \langle x, A\mathbf{v} \rangle,
\end{aligned}$$

where $\tilde{x}$ is any representative of x in $\mathbb{R}^2$, $y \cdot z$ denotes the Euclidean inner product for $y, z \in \mathbb{R}^2$, and the last equality holds because A is symmetric. This shows that

$$f \cdot \mathbf{v} = \sum_{k \in \mathbb{Z}} c_f(k) A^k \mathbf{v}$$

for any $f = \sum_{k \in \mathbb{Z}} c_f(k) u^k \in R_1$ and any $\mathbf{v} \in \mathbb{Z}^2$.

Example 3.6 (Square Shift VI)**.** Recall that the Square Shift is given by the subshift σ of $\mathbb{Z}^2$ on the space

$$X = \{(x_{\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{\mathbf{m}} + x_{\mathbf{m}+(1,0)} + x_{\mathbf{m}+(0,1)} + x_{\mathbf{m}+(1,1)} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^2\}.$$

We first observe that

$$\widehat{(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2},$$

where we used the fact that the Pontryagin dual of a finite cyclic group is isomorphic to itself and that the dual of a product of compact abelian groups may be identified with the direct sum of the duals (see Propositions A.8 and A.10). Given $x = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ and $\chi = (\chi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}$, we further observe that

$$\langle x, \chi \rangle = \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^2} \frac{x_{\mathbf{n}} \chi_{\mathbf{n}}}{2}\right),$$

where in the sum we identify $(\mathbb{Z}/2\mathbb{Z})$ with $\{0, 1\} \subseteq \mathbb{R}$. For $\mathbf{m} \in \mathbb{Z}^2$, we consider $\chi^{(\mathbf{m})} = (\chi_{\mathbf{n}}^{(\mathbf{m})})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}$ defined by

$$\chi_{\mathbf{n}}^{(\mathbf{m})} = \begin{cases} 1 & \text{if } \mathbf{n} - \mathbf{m} \in \{(0,0), (0,1), (1,0), (1,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

for all $\mathbf{n} \in \mathbb{Z}^2$. Then, the condition $x_{\mathbf{m}} + x_{\mathbf{m}+(1,0)} + x_{\mathbf{m}+(0,1)} + x_{\mathbf{m}+(1,1)} = 0$ for all $\mathbf{m} \in \mathbb{Z}^2$ can equivalently be stated as $\langle x, \chi^{(\mathbf{m})} \rangle = 1$ for all $\mathbf{m} \in \mathbb{Z}^2$. Thus, if we set $H \subseteq (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}$ to be the subgroup generated by the elements $\chi^{(\mathbf{m})}$, then X is given by the annihilator $H^{\perp}$ of H and we obtain that $\widehat{X} = \widehat{H^{\perp}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}/H$. Finally, we wish to determine the scalar multiplication of R_2 on $\widehat{X}$, i.e. the dual action $\hat{\sigma}$. Let $x = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in X$ and $\eta = (\eta_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}/H$, then

$$\begin{aligned} \langle x, \hat{\sigma}_{\mathbf{m}}(\eta) \rangle &= \langle \sigma_{\mathbf{m}}(x), \eta \rangle \\ &= \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^2} \frac{x_{\mathbf{n}+\mathbf{m}} \eta_{\mathbf{n}}}{2}\right) \\ &= \exp\left(2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^2} \frac{x_{\mathbf{n}} \eta_{\mathbf{n}-\mathbf{m}}}{2}\right), \end{aligned}$$

for any $\mathbf{m} \in \mathbb{Z}^2$. Thus, we see that $\hat{\sigma}_{\mathbf{m}}((\eta_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2}) = (\eta_{\mathbf{n}-\mathbf{m}})_{\mathbf{n} \in \mathbb{Z}^2}$ for all $\mathbf{m} \in \mathbb{Z}^2$, i.e., the dual action is a "reversed" shift-action on $(\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}/H$. To verify that this action is well-defined on the quotient, we observe that the subgroup H is indeed invariant under this "reversed" shift-action lifted to $(\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}^2}$. Specifically, for any generator $\chi^{(\mathbf{m})}$ of H and any $\mathbf{n} \in \mathbb{Z}^2$, the "reversed" shift-action at $\mathbf{n}$ maps $\chi^{(\mathbf{m})}$ to $\chi^{(\mathbf{m}+\mathbf{n})}$, which remains in H.

In both preceding examples, we arrived at an explicit description of the dual module via a direct computation of the Pontryagin dual and of the dual action. We will later see that, if possible, expressing the dual modules in terms of quotients of the form $R_d/\mathfrak{a}$ for an ideal $\mathfrak{a} \subseteq R_d$ is a more useful description. In the case of Arnold's Cat Map and the Square Shift this is possible, that is, their dual modules are cyclic. In fact, the following two examples illustrate the wide variety of dynamical systems whose dual module is cyclic.

Example 3.7 (Dual Modules of Toral Automorphisms)**.** Let $B \in \mathrm{GL}_d(\mathbb{Z})$ be an invertible integer matrix and consider the corresponding toral automorphism $T_B : \mathbb{T}^d \to \mathbb{T}^d$. Our goal is to describe the dual module of the algebraic $\mathbb{Z}$-action T_B. As alluded to above, we would ideally like to describe it as a quotient $R_1/\mathfrak{a}$. It turns out that this is not possible for all toral automorphisms.

Let $f(u) = \sum_{j=0}^{d} c_j u^j$ denote the (monic) characteristic polynomial of the matrix B and consider the companion matrix

$$C_f = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{d-1} \end{pmatrix}$$

of f. The case where the dual module of T_B can be expressed as a quotient $R_1/\mathfrak{a}$ corresponds to the case where the matrix B is conjugate to the companion matrix C_f over $\mathbb{Z}$, i.e., there exists a matrix $P \in \mathrm{GL}_d(\mathbb{Z})$ such that $B = PC_fP^{-1}$.[3] We henceforth assume that B is conjugate to C_f over $\mathbb{Z}$. In this case, the toral automorphism T_P induced by the matrix P satisfies $T_B^k \circ T_P = T_P \circ T_{C_f}^k$ for all integers k. This shows that the algebraic $\mathbb{Z}$-actions induced by T_B and T_{C_f} are conjugate.

We claim that the algebraic $\mathbb{Z}$-action T_{C_f} (and thus T_B) is conjugate to the algebraic $\mathbb{Z}$-action corresponding to the R_1-module $R_1/(f)$ via Proposition 3.2, or equivalently, that the dual module of T_B is isomorphic to $R_1/(f)$. First, notice that $X^{R_1/(f)} = \widehat{R_1/(f)}$ is isomorphic to the annihilator $(f)^{\perp}$ (by Proposition A.7). Thus, by Example 3.4 and Proposition 3.3, we may identify $(X^{R_1/(f)}, \alpha^{R_1/(f)})$ with the subshift on the closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$ given by

$$\left\{ (x_n) \in \mathbb{T}^{\mathbb{Z}} : \sum_{i=0}^{d} c_i x_{i+m} = 0 \text{ for all } m \in \mathbb{Z} \right\}. \tag{3.1}$$

Now define

$$\begin{aligned} \varphi : X^{R_1/(f)} &\to \mathbb{T}^d \\ (x_n) &\mapsto (x_0, \dots, x_{d-1}) \end{aligned}$$

and notice that by the defining relation in Eq. (3.1), φ is a continuous group isomorphism.

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[3] This dichotomy is related to the ideal class group of the order $\mathbb{Z}[u]/(f(u))$ via the Latimer–MacDuffee theorem.

Furthermore, for any $x = (x_n) \in X^{R_1/(f)}$ we have

$$\begin{aligned} T_{C_f} \circ \varphi(x) &= T_{C_f}(x_0, \dots, x_{d-1}) \\ &= \left(x_1, \dots, x_{d-1}, -\sum_{i=0}^{d-1} c_i x_i\right) \\ &= (x_1, \dots, x_{d-1}, x_d) \\ &= \varphi \circ \alpha^{R_1/(f)}(x), \end{aligned}$$

where we used that $\alpha^{R_1/(f)}$ is given by the shift action. This proves our claim.

Example 3.8 (Symbolic Shift-Actions via Cyclic Modules)**.** Let $\mathfrak{a} = (k, f)$ be the ideal generated by an integer $k \geq 2$ and a polynomial $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in R_d$ for an integer $d \geq 1$. We set $M = R_d/\mathfrak{a}$ and view it as an R_d-module. Via Proposition 3.2, we obtain the algebraic $\mathbb{Z}^d$-action $(X, \alpha) = (X^{R_d/\mathfrak{a}}, \alpha^{R_d/\mathfrak{a}})$. By Pontryagin duality, $X = \widehat{R_d/\mathfrak{a}}$ is isomorphic to the annihilator $\mathfrak{a}^{\perp}$. Recalling from Example 3.4 that the algebraic $\mathbb{Z}^d$-action corresponding to R_d is the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$, Proposition 3.3 implies that (X, α) may be identified with a subshift of $\mathbb{T}^{\mathbb{Z}^d}$. Concretely, we have

$$\begin{aligned} X &\cong \left\{x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : \langle x, g \rangle = 1 \text{ for all } g \in \mathfrak{a}\right\} \\ &\cong \left\{x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : k \cdot x = 0 \text{ and } \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}+\mathbf{m}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d\right\} \\ &\cong \left\{x = (x_{\mathbf{n}}) \in (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}+\mathbf{m}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d\right\} \end{aligned}$$

and $\alpha_{\mathbf{m}}((x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}) = (x_{\mathbf{n}+\mathbf{m}})_{\mathbf{n} \in \mathbb{Z}^d}$ for any $(x_{\mathbf{n}}) \in (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}^d}$ and any $\mathbf{m} \in \mathbb{Z}^d$. We see that the integer $k \in \mathfrak{a}$ forces the subshift to have a finite alphabet, $\mathbb{Z}/k\mathbb{Z}$, while the polynomial f specifies a *local* rule.[4] This suggests that (X, α) is a subshift of finite type (cf. Definition 2.10). Indeed, since f is a polynomial, there is a finite subset $F \subseteq \mathbb{Z}^d$ such that $c_f(\mathbf{n}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d \smallsetminus F$. Then setting

$$A = \left\{(x_{\mathbf{n}}) \in (\mathbb{Z}/k\mathbb{Z})^F : \sum_{\mathbf{n} \in F} c_f(\mathbf{n}) x_{\mathbf{n}} = 0\right\},$$

we have

$$X \cong \left\{x \in (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}^d} : \pi_F(\alpha_{\mathbf{m}}(x)) \in A \text{ for all } \mathbf{m} \in \mathbb{Z}^d\right\},$$

where $\pi_F : (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}^d} \to (\mathbb{Z}/k\mathbb{Z})^F$ denotes the canonical projection.

We thus conclude that (X, α), the algebraic $\mathbb{Z}^d$-action dual to the cyclic R_d-module $R_d/(k, f)$, is conjugate to a subshift of $(\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}^d}$ of finite type, where the local rule is determined by the polynomial f.

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[4] One could also consider the case $k = 0$ yielding a subshift of $\mathbb{T}^{\mathbb{Z}^d}$ without any finiteness restrictions on the alphabet. In this case, f still specifies a local rule.

3.2 The Spectrum of an Algebraic Action

The goal of this section is to make the notion of the “spectrum” of an algebraic $\mathbb{Z}^d$-action concrete. We will achieve this via the correspondence between the class of algebraic $\mathbb{Z}^d$-actions and the class of countable R_d-modules established in the previous section.

Let $B \in \mathrm{GL}_n(\mathbb{Z})$ be an invertible integer matrix and $T_B : \mathbb{T}^n \to \mathbb{T}^n$ the corresponding toral automorphism. We have previously observed that ergodicity (equivalently, topological transitivity, by Corollary 2.16), mixing, and expansiveness are encoded in the spectrum of the matrix B. If we denote by f_B the characteristic polynomial of B, then the spectrum of B is given by

$$\operatorname{spec} B = \{\lambda \in \mathbb{C}^\times : f_B(\lambda) = 0\},$$

where we implicitly used the fact that T_B is an automorphism and thus the matrix B has no zero eigenvalues. Recall from Example 3.7 that the dual module of T_B is isomorphic to $R_1/(f_B)$. This motivates the following general definition.

Definition 3.9. Let $d \geq 1$ be an integer. For a prime number p, we denote by $\overline{\mathbb{F}}_p$ an algebraic closure of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. For notational consistency, we write $\overline{\mathbb{F}}_0$ for an algebraic closure of $\mathbb{Q}$. For a prime ideal $\mathfrak{p} \subseteq R_d$, we define the algebraic variety

$$\mathrm{V}(\mathfrak{p}) = \{(c_1, \dots, c_d) \in (\overline{\mathbb{F}}^\times_{\mathrm{char}(R_d/\mathfrak{p})})^d : f(c_1, \dots, c_d) = 0 \text{ for every } f \in \mathfrak{p}\},$$

and for any ideal $\mathfrak{a} \subseteq R_d$, the complex variety

$$\mathrm{V}_\mathbb{C}(\mathfrak{a}) = \{(c_1, \dots, c_d) \in (\mathbb{C}^\times)^d : f(c_1, \dots, c_d) = 0 \text{ for every } f \in \mathfrak{a}\}.$$

The algebraic variety $\mathrm{V}(\mathfrak{p})$ and the complex variety $\mathrm{V}_\mathbb{C}(\mathfrak{a})$ play distinct roles. The definition of $\mathrm{V}(\mathfrak{p})$ respects the characteristic of $R_d/\mathfrak{p}$ and remains meaningful in all cases. By contrast, the complex variety disregards the characteristic and always lies in $(\mathbb{C}^\times)^d$. While this allows us to leverage the rich metric structure inherent to $\mathbb{C}$, it also means that if the characteristic of $R_d/\mathfrak{p}$ is positive, then $\mathrm{V}_\mathbb{C}(\mathfrak{p})$ is empty.

Nevertheless, both notions will be useful. As we will see, while many dynamical properties are fully encoded in the algebraic variety $\mathrm{V}(\mathfrak{p})$, expansiveness is an intrinsically topological property that relies on metric information. Recall, for example, that for a toral automorphism, expansiveness is characterized by the absence of eigenvalues of modulus 1. This condition explicitly depends on the absolute value on $\mathbb{C}$ and is not intrinsic to the purely algebraic setting of $\mathrm{V}(\mathfrak{p})$. We finally note that although one could technically circumvent the use of complex varieties to characterize expansiveness by fixing an embedding of $\overline{\mathbb{F}}_0$ into $\mathbb{C}$, we introduce them explicitly here as they will appear in the proofs in any case.

We return to our toral automorphism $T_B : \mathbb{T}^n \to \mathbb{T}^n$. Note that R_d is a unique factorization domain for any integer $d \geq 1$. This can be seen, for example, by viewing R_d as a localization of the unique factorization domain $\mathbb{Z}[u_1, \dots, u_d]$ at the multiplicative set generated by $u_1, \dots, u_d$. Thus, we may consider a decomposition $f_B = p_1^{e_1} \cdots p_k^{e_k}$ of f_B into primes $p_j \in R_1$. In pursuit of our goal to define the “spectrum” of an algebraic $\mathbb{Z}^d$-action, i.e., of a countable R_d-module, we ask how the prime ideals $(p_1), \dots, (p_k)$, which together determine the spectrum

$$\operatorname{spec} B = \{\lambda \in \mathbb{C}^\times : f_B(\lambda) = 0\} = \bigcup_{j=1}^{k} \mathrm{V}_\mathbb{C}((p_j)),$$

are related to the R_1-module $R_1/(f_B)$. It turns out that the prime ideals $(p_1), \dots, (p_k)$ are exactly given by the set of primes associated to $R_1/(f_B)$, i.e., $\mathrm{Ass}(R_1/(f_B)) = \{(p_1), \dots, (p_k)\}$. Indeed, since R_1 is a unique factorization domain, the intersection $(f_B) = (p_1^{e_1}) \cap \cdots \cap (p_k^{e_k})$ is a minimal primary decomposition and hence we obtain that $(p_1), \dots, (p_k)$ are exactly the primes associated to $R_1/(f_B)$ (see Corollary C.19).

To further motivate the central role of the associated prime ideals, we consider an R_d-module M. For pedagogical reasons, we assume that M is Noetherian. In this case, M admits a prime filtration (see Proposition C.13), that is, there exists a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

and a sequence of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of R_d such that the quotient M_j/M_{j-1} is isomorphic to $R_d/\mathfrak{p}_j$ for each $j = 1, \dots, n$. Using this filtration, we may express M as being composed of the quotients $R_d/\mathfrak{p}_j$ in the following sense. For each $j = 1, \dots, n$, we have the short exact sequence

$$0 \to M_{j-1} \to M_j \to R_d/\mathfrak{p}_j \to 0$$

so that M_j may be viewed as an extension of M_{j-1} by $R_d/\mathfrak{p}_j$. In a sense, this reduces the problem of understanding M to understanding the quotients $R_d/\mathfrak{p}_j$ for $j = 1, \dots, n$. Crucially, the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are not arbitrary. They are closely related to the set $\mathrm{Ass}(M)$ of primes associated to M. Indeed, we have $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, and for each $j = 1, \dots, n$, there exists a $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\mathfrak{p} \subseteq \mathfrak{p}_j$. In this sense, the primes associated to M determine the building blocks of the module M.

To further illustrate the importance of this prime filtration, and thus of the associated primes, for our discussion, we observe that a short exact sequence of R_d-modules

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

induces via Proposition 3.2 a short exact sequence of algebraic $\mathbb{Z}^d$-actions

$$0 \to X^C \xrightarrow{\hat{\psi}} X^B \xrightarrow{\hat{\varphi}} X^A \to 0,$$

that is, a short exact sequence of compact metrizable abelian groups such that $\alpha^B_{\mathbf{n}} \circ \hat{\psi} = \hat{\psi} \circ \alpha^C_{\mathbf{n}}$ and $\alpha^A_{\mathbf{n}} \circ \hat{\varphi} = \hat{\varphi} \circ \alpha^B_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$. Thus, we may view X^C as a closed, α^B-invariant subgroup of X^B and X^A as the quotient X^B/X^C. Consequently, Lemma 1.29 and Proposition 2.15 suggest that if we understand the dynamics of the building blocks X^C and X^A, we gain an understanding of the dynamics of X^B.

To make this concrete, let M again be a Noetherian R_d-module with a prime filtration as above. Assume that for every j, the algebraic $\mathbb{Z}^d$-action $(X^{R_d/\mathfrak{p}_j}, \alpha^{R_d/\mathfrak{p}_j})$ is ergodic. For $j = 1$, we have $M_1 \cong R_d/\mathfrak{p}_1$, so α^{M_1} is ergodic. For $j = 2$, the filtration yields the short exact sequence

$$0 \to X^{R_d/\mathfrak{p}_2} \to X^{M_2} \to X^{M_1} \to 0.$$

Since both the subsystem $\alpha^{R_d/\mathfrak{p}_2}$ and the quotient system α^{M_1} are ergodic, Proposition 2.15 implies that α^{M_2} is ergodic. Proceeding inductively via the short exact sequences

$$0 \to X^{R_d/\mathfrak{p}_j} \to X^{M_j} \to X^{M_{j-1}} \to 0,$$

we conclude that $\alpha^M = \alpha^{M_n}$ is ergodic. The same argument shows that if the "building blocks" $(X^{R_d/\mathfrak{p}_j}, \alpha^{R_d/\mathfrak{p}_j})$ are mixing or expansive, then α^M is mixing or expansive, respectively.

All of the above motivates the informal definition of the "spectrum" of an algebraic $\mathbb{Z}^d$-action (X, α) as the collection of varieties of the prime ideals associated to the dual module $\widehat{X}$.

3.3 Algebraic Characterizations of Dynamical Properties

In this section, we will use the correspondence between algebraic $\mathbb{Z}^d$-actions and countable R_d-modules introduced in Section 3.1 and the notion of the "spectrum" of an algebraic $\mathbb{Z}^d$-action introduced in Section 3.2 to establish a dictionary between dynamical properties of such actions and the algebraic properties of the dual module.

3.3.1 Structural Finiteness & Periodic Points

We begin our application of the correspondence established in Proposition 3.2 by revisiting the descending chain condition. We will characterize it in terms of purely algebraic conditions on the dual module and show that it implies the density of periodic points.

Proposition 3.10. *Let (X, α) be an algebraic $\mathbb{Z}^d$-action for $d \geq 1$. Then the following are equivalent:*

(a) The action α satisfies the descending chain condition.
(b) The R_d-module $M = \widehat{X}$ obtained via Proposition 3.2 is Noetherian.
(c) The action α is conjugate to a subshift of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some integer $n \geq 1$.

Proof. (b) $\implies$ (c): Since M is Noetherian, there exists a surjective R_d-module homomorphism $R_d^n \to M$ for some integer $n \geq 1$. This induces an injective and continuous dual homomorphism $X \to X^{R_d^n}$ (see Proposition A.11). Following Example 3.4, we see that $X^{R_d^n}$ is conjugate to the shift-action of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$, making X conjugate to a subshift of $(\mathbb{T}^n)^{\mathbb{Z}^d}$.

(c) $\implies$ (a): This is an immediate consequence of Theorem 2.8.

(a) $\implies$ (b): Let $M_1 \subseteq M_2 \subseteq \ldots$ be a non-decreasing sequence of submodules of M. Then by Proposition 3.3 this sequence corresponds to a non-increasing sequence of closed, α-invariant subgroups of X. By our assumption, this sequence eventually stabilizes, hence so does the sequence of submodules. ☐

Theorem 3.11. *Let (X, α) be an algebraic $\mathbb{Z}^d$-action satisfying the descending chain condition. Then the subgroup of α-periodic points is dense in X.*

Proof. Let $M = \widehat{X}$ be the dual R_d-module of (X, α) obtained via Proposition 3.2. By Proposition 3.10, we know that M is Noetherian. Thus, for each non-zero element $a \in M$, there exists a maximal submodule $M_a \subseteq M$ with the property that $a \notin M_a$. We claim that M/M_a is finite for every non-zero $a \in M$. Assuming this claim for now, we obtain that the α-invariant subgroup $X_a = M_a^{\perp}$ of X is finite and not annihilated by a (by Propositions A.7 and 3.3). Thus, each point in X_a is α-periodic. This shows that no non-zero character of X vanishes on the subgroup P of α-periodic points of X, i.e., $P^{\perp} = \{0\} \subseteq M$. By continuity of the characters, we have $P^{\perp} = \overline{P}^{\perp}$, where $\overline{P}$ denotes the closure of P in X, which is again an α-invariant subgroup (Proposition A.1). Since we also have $X^{\perp} = \{0\}$, we would be able to conclude the proof using Proposition 3.3.

It thus only remains to prove the claim. Fix a non-zero element $a \in M$ and set $N = M/M_a$. By maximality of M_a, we obtain that $\tilde{N} = (R_d \cdot a + M_a)/M_a$ is the unique minimal non-zero submodule of N. We wish to show that the annihilator $\mathfrak{a} = \mathrm{Ann}_{R_d}(\tilde{N})$ is a maximal ideal in R_d. Assume, for a contradiction, that there exists an ideal $\mathfrak{b}$ of R_d such that $\mathfrak{a} \subsetneq \mathfrak{b} \subsetneq R_d$. Then by minimality of $\tilde{N}$, we have that $\mathfrak{b} \cdot \tilde{N} = \tilde{N}$. Thus, Nakayama's lemma (Lemma C.5) implies the existence of an element $b \in \mathfrak{b}$ such that $(1 + b) \cdot \tilde{N} = 0$. This yields $1 + b \in \mathfrak{a} \subseteq \mathfrak{b}$, so $1 \in \mathfrak{b}$, a contradiction. We have thus proven that $R_d/\mathfrak{a}$ is a field.

Our next goal is to show that $R_d/\mathfrak{a}$ is finite. First, notice that the contraction $\mathbb{Z} \cap \mathfrak{a}$ is a prime ideal $\mathfrak{p}$ of $\mathbb{Z}$. Thus, we have an embedding $\mathbb{Z}/\mathfrak{p} \hookrightarrow R_d/\mathfrak{a}$. Since R_d is a finitely generated $\mathbb{Z}$-algebra, we see that $R_d/\mathfrak{a}$ is a finitely generated $\mathbb{Z}/\mathfrak{p}$-algebra.

This implies that $\mathfrak{p}$ cannot be the zero ideal. Indeed, suppose $\mathfrak{p} = (0)$. Then $R_d/\mathfrak{a}$ is a finitely generated $\mathbb{Q}$-algebra and Zariski's lemma (Lemma C.8) implies that $R_d/\mathfrak{a}$ is, in fact, a finitely generated $\mathbb{Q}$-module. Now, the Artin–Tate lemma (Lemma C.6) implies that $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-algebra. This can be seen to be impossible, for example, by considering the fractions $1/p \in \mathbb{Q}$ for every prime number p. Thus, $\mathfrak{p}$ is not the zero ideal.

So we have that $R_d/\mathfrak{a}$ is a finitely generated algebra over the finite field $\mathbb{Z}/\mathfrak{p}$. A final application of Zariski's lemma (Lemma C.8) implies that $R_d/\mathfrak{a}$ is a finite field extension of $\mathbb{Z}/\mathfrak{p}$, showing that $R_d/\mathfrak{a}$ is finite.

Next, we claim that there exists an integer $m \geq 1$ such that $\mathfrak{a}^m \cdot N = 0$. Assume, for a contradiction, that for all integers $m \geq 1$ we have $a + M_a \in \mathfrak{a}^m \cdot N$. This implies that $L = \bigcap_{m \geq 1} \mathfrak{a}^m \cdot N$ is non-zero since $a + M_a \in L$. Because R_d is Noetherian and N is finitely generated, the Artin–Rees lemma (Lemma C.7) implies that $\mathfrak{a} \cdot L = L$. Nakayama's lemma (Lemma C.5) then yields an element $y \in 1 + \mathfrak{a}$ such that $y \cdot L = 0$. We observe that $\tilde{N} \subseteq L$, because L is non-zero. This implies that y also annihilates $\tilde{N}$, i.e., $y \in \mathfrak{a} = \mathrm{Ann}_{R_d}(\tilde{N})$, which in turn gives $1 \in \mathfrak{a}$. This is a contradiction and thus there exists an integer $m \geq 1$ such that $a + M_a \notin \mathfrak{a}^m \cdot N$. Now minimality of $\tilde{N} = (R_d \cdot a + M_a)/M_a$ implies that $\mathfrak{a}^m \cdot N = 0$.

We thus have the finite non-increasing sequence $N \supseteq \mathfrak{a} \cdot N \supseteq \cdots \supseteq \mathfrak{a}^m \cdot N = 0$, where each quotient $\mathfrak{a}^j \cdot N / \mathfrak{a}^{j+1} \cdot N$ may be viewed as a Noetherian module over $R_d/\mathfrak{a}$, i.e., a finite-dimensional vector space over the field $R_d/\mathfrak{a}$. Since $R_d/\mathfrak{a}$ is finite, each quotient is finite. By induction, we may now conclude that N is finite. ☐

3.3.2 Ergodicity & Mixing

We now turn to translating the properties of ergodicity and mixing into the algebraic world by using the spectral characterizations developed in Section 2.2.

Theorem 3.12. *Let M be a countable R_d-module and $(X, \alpha) = (X^M, \alpha^M)$ as obtained from Proposition 3.2. Then the following are equivalent:*

- *(a) α is ergodic.*
- *(b) $\alpha^{R_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated to M.*
- *(c) No prime ideal associated to M contains a set of the form $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ for an integer $l \geq 1$.*

Moreover, if M is Noetherian, then there are only finitely many primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ associated to M and the above are further equivalent to the following.

- *(d) $\alpha_{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$.*
- *(e) For every $i \in \{1, \ldots, m\}$ and every integer $l \geq 1$ we have*

$$\mathrm{V}(\mathfrak{p}_i) \not\subseteq \{(c_1, \ldots, c_d) \in (\overline{\mathbb{F}}^{\times}_{\mathrm{char}(R_d/\mathfrak{p}_i)})^d : c_1^l = \cdots = c_d^l = 1\}.$$

Remark 3.13. If $\alpha : \Gamma \to \mathrm{Aut}(X)$ is a quasi-algebraic action, then we say that the single automorphism α_γ, for some $\gamma \in \Gamma$, is topologically transitive, ergodic, mixing, or expansive if the quasi-algebraic $\mathbb{Z}$-action $k \mapsto \alpha_{\gamma^k}$ is topologically transitive, ergodic, mixing, or expansive, respectively.

We begin with a useful lemma, which, in essence, is a careful interpretation of Theorem 2.13 in the specific case of an algebraic $\mathbb{Z}^d$-action.

Lemma 3.14. *Let M be a countable R_d-module and $(X, \alpha) = (X^M, \alpha^M)$ as obtained from Proposition 3.2. Then the following are equivalent for any $\mathbf{n} \in \mathbb{Z}^d$:*

(a) $\alpha_{\mathbf{n}}$ is ergodic.
(b) $\alpha_{\mathbf{n}}^{R_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated to M.
(c) No prime ideal associated to M contains a polynomial of the form $u^{l\mathbf{n}} - 1$ for any integer $l \geq 1$.

Proof. (a) $\iff$ (c): We first note that $M = \widehat{X}$ corresponds to the set of all irreducible unitary representations of X up to unitary equivalence. Thus, applying Theorem 2.13 to the algebraic $\mathbb{Z}$-action $k \mapsto \alpha_{k\mathbf{n}}$ yields that $\alpha_{\mathbf{n}}$ is non-ergodic if and only if there exists a non-zero $a \in M$ such that the stabilizer of a under the action of $\mathbb{Z}$ generated by $\hat{\alpha}_{\mathbf{n}}$ has finite index in $\mathbb{Z}$. Recalling that $\hat{\alpha}_{k\mathbf{n}}(a) = u^{k\mathbf{n}} \cdot a$, the latter is in turn equivalent to the existence of a non-zero $a \in M$ and an integer $l \geq 1$ such that $(u^{l\mathbf{n}} - 1) \cdot a = 0$, i.e., $u^{l\mathbf{n}} - 1$ lies in the annihilator $\operatorname{Ann}(a)$ of a. Given this equivalence, the following observation concludes this part of the proof.

Let a be an arbitrary non-zero element of M and consider the non-trivial submodule $N = R_d \cdot a$ of M. Since R_d is Noetherian, there exists a prime ideal $\mathfrak{p}$ of R_d associated to N, i.e., $\mathfrak{p} = \operatorname{Ann}(b)$ for some $b \in N$ (see Proposition C.14). Since by construction we have $b = f \cdot a$ for some $f \in R_d$, we see that $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b) = \mathfrak{p}$.

(b) $\iff$ (c): Let $\mathfrak{p}$ be a prime ideal associated to M and notice that $R_d/\mathfrak{p}$ is a countable R_d-module, whose only associated prime is $\mathfrak{p}$ itself. Thus, applying the equivalence of (a) and (c) to the module $R_d/\mathfrak{p}$ for each prime $\mathfrak{p}$ associated to M, concludes the proof. ☐

Proof of Theorem 3.12. The proof of the first set of equivalences is very similar to the proof of Lemma 3.14.

(a) $\iff$ (c): Theorem 2.13 yields that α is non-ergodic if and only if there exists a non-zero $a \in M$ such that the stabilizer $\{\mathbf{n} \in \mathbb{Z}^d : (u^{\mathbf{n}} - 1) \cdot a = 0\}$ has finite index in $\mathbb{Z}^d$. We note that a subgroup of $\mathbb{Z}^d$ has finite index if and only if it contains $l\mathbb{Z}^d$ for some integer $l \geq 1$. Thus, α is non-ergodic if and only if there exists a non-zero $a \in M$ and an integer $l \geq 1$ such that $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ is contained in $\operatorname{Ann}(a)$. By the observation made in the proof of Lemma 3.14 regarding the associated prime ideals, we are done.

(b) $\iff$ (c): Again, we apply the equivalence of (a) and (c) to the countable R_d-modules $R_d/\mathfrak{p}$ whose only associated prime is $\mathfrak{p}$, and we do this for all primes $\mathfrak{p}$ associated to M.

(d) $\Longrightarrow$ (a): This follows immediately from the definitions, as any α-invariant set is also $\alpha_{\mathbf{n}}$-invariant for any $\mathbf{n} \in \mathbb{Z}^d$.

(c) $\Longrightarrow$ (d): We prove the contrapositive, i.e., assume that $\alpha_{\mathbf{n}}$ is non-ergodic for every $\mathbf{n} \in \mathbb{Z}^d$. There are only finitely many primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ associated to M, since it is Noetherian (Proposition C.14). Thus, our assumption and Lemma 3.14 imply that for every $\mathbf{n} \in \mathbb{Z}^d$ there exists an integer $l \geq 1$ such that $u^{l\mathbf{n}} - 1 \in \bigcup_{i=1}^m \mathfrak{p}_i$.

We define the sets $\Gamma_i = \{\mathbf{n} \in \mathbb{Z}^d : u^{\mathbf{n}} - 1 \in \mathfrak{p}_i\}$, for $i = 1, \ldots, m$. It is readily checked that these sets form subgroups of $\mathbb{Z}^d$ by using the identities $u^{\mathbf{n}+\mathbf{m}} - 1 = u^{\mathbf{n}}(u^{\mathbf{m}} - 1) + (u^{\mathbf{n}} - 1)$ and $u^{-\mathbf{n}} - 1 = -u^{-\mathbf{n}}(u^{\mathbf{n}} - 1)$.

Assume, for a contradiction, that each Γ_i has infinite index. Then each Γ_i spans a subspace $V_i = \operatorname{span}_{\mathbb{Q}}(\Gamma_i)$ of dimension strictly less than d in $\mathbb{Q}^d$. Since a vector space over an infinite field cannot be written as a finite union of proper subspaces, the union of the subspaces V_i cannot be the whole space $\mathbb{Q}^d$. Thus, there exists a non-zero vector in $\mathbb{Q}^d \setminus \bigcup_{i=1}^m V_i$. By multiplying out the denominator, we remain outside of the union of the subspaces and thus obtain a non-zero $\mathbf{n} \in \mathbb{Z}^d$ that is not contained in any of the V_i's. Since V_i is a vector space, $l\mathbf{n} \in V_i$ would imply $\mathbf{n} \in V_i$. Thus, for every integer $l \geq 1$ we have $l\mathbf{n} \notin \Gamma_i$, i.e., $u^{l\mathbf{n}} - 1 \notin \mathfrak{p}_i$, for all $i = 1, \ldots, m$. This is a contradiction to our conclusion above and we therefore conclude that Γ_i has finite index in $\mathbb{Z}^d$ for some i. Hence, there exists an $l \geq 1$ such that $l\mathbb{Z}^d \subseteq \Gamma_i$, i.e., $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \subseteq \mathfrak{p}_i$.

(c) $\iff$ (e): We prove that the negations imply each other. The proof is essentially an application of Hilbert's Nullstellensatz. Let $\mathfrak{p} \subseteq R_d$ be a prime ideal, and set $p = \operatorname{char}(R_d/\mathfrak{p})$, $K = \overline{\mathbb{F}}_p$ and $\mu_l = \{x \in K : x^l = 1\}$ for any integer $l \geq 1$. Then, it is enough to show that for every integer $l \geq 1$, we have $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \subseteq \mathfrak{p}$ if and only if $\mathrm{V}(\mathfrak{p}) \subseteq \mu_l^d$.

We observe that the containment $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \subseteq \mathfrak{p}$ is equivalent to $u_j^l - 1 \in \mathfrak{p}$ for every $j = 1, \ldots, d$. This may be seen using the identities $uv - 1 = v(u-1) + (v-1)$ and $u^{-1} - 1 = -u^{-1}(u-1)$.

First, assume that there is an integer $l \geq 1$ such that $u_j^l - 1 \in \mathfrak{p}$ for all $j = 1, \ldots, d$. Then, for any point $(c_1, \ldots, c_d) \in \mathrm{V}(\mathfrak{p})$ we must have $c_j^l - 1 = 0$ for every $j = 1, \ldots, d$. Hence, we conclude $\mathrm{V}(\mathfrak{p}) \subseteq \mu_l^d$.

Second, assume that there is an integer $l \geq 1$ such that $\mathrm{V}(\mathfrak{p}) \subseteq \mu_l^d$. This direction requires more work. Set $F = \mathbb{Q}$ if $p = 0$ and $F = \mathbb{F}_p$ if $p > 0$. Notice that in both cases, we have a homomorphism from R_d to $F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, either induced by the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ or by the epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$. Let $\mathfrak{a}$ denote the extension of $\mathfrak{p}$ to $F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, that is, the ideal generated by the image of $\mathfrak{p}$ in $F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$.

Since K is a field extension of F, it is in particular a vector space over F. We may choose an F-basis $\{e_i\}_{i \in I}$ of K with $e_{i_0} = 1$ for some $i_0 \in I$. This yields the direct sum decomposition $K[u_1^{\pm1}, \ldots, u_d^{\pm1}] = \bigoplus_{i \in I} F[u_1^{\pm1}, \ldots, u_d^{\pm1}] \cdot e_i$. Let $\mathfrak{b}$ denote the extension of $\mathfrak{a}$ in $K[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, i.e., $\mathfrak{b} = \mathfrak{a} \cdot K[u_1^{\pm1}, \ldots, u_d^{\pm1}]$. By expanding any element of $\mathfrak{b}$ in the basis above, we see that $\mathfrak{b} = \bigoplus_{i \in I} \mathfrak{a} \cdot e_i$. Now, if $f \in \mathfrak{b} \cap F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, then f can be written both as $f \cdot e_{i_0}$ (with all other components zero) and as $\sum_i a_i e_i$ with each $a_i \in \mathfrak{a}$. By uniqueness of representation, we obtain $f = a_{i_0} \in \mathfrak{a}$. Thus $\mathfrak{b} \cap F[u_1^{\pm1}, \ldots, u_d^{\pm1}] = \mathfrak{a}$.

For an ideal $\mathfrak{c} \subseteq K[u_1^{\pm1}, \ldots, u_d^{\pm1}]$ we set (analogously to Definition 3.9)

$$\mathrm{V}(\mathfrak{c}) = \{(c_1, \ldots, c_d) \in (K^\times)^d : f(c_1, \ldots, c_d) = 0 \text{ for all } f \in \mathfrak{c}\}.$$

Let $\mathfrak{q}$ denote the ideal in $K[u_1^{\pm1}, \ldots, u_d^{\pm1}]$ generated by the monomials $u_j^l - 1$ for $j = 1, \ldots, d$. Then we have $\mathrm{V}(\mathfrak{q}) = \mu_l^d$ and our assumption thus translates to $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(\mathfrak{q})$. Since $\mathfrak{b}$ is generated by the image of $\mathfrak{p}$ in $K[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, any point in $(K^\times)^d$ vanishing on $\mathfrak{b}$ must also vanish on the image of $\mathfrak{p}$. Thus, we see that $\mathrm{V}(\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{q})$. Applying Hilbert's Nullstellensatz via the identification

$$K[u_1^{\pm1}, \ldots, u_d^{\pm1}] \cong K[u_1, v_1, \ldots, u_d, v_d]/(u_1 v_1 - 1, \ldots, u_d v_d - 1),$$

we obtain that $\operatorname{rad} \mathfrak{q} \subseteq \operatorname{rad} \mathfrak{b}$. This implies that there is an integer $N \geq 1$ such that $(u_j^l - 1)^N \in \mathfrak{b}$ for all $j = 1, \ldots, d$. Since each $(u_j^l - 1)^N$ lies in $F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, we have $(u_j^l - 1)^N \in \mathfrak{a}$ for all j.

If $p = 0$, then $\mathfrak{a} = \mathfrak{p}\mathbb{Q}[u_1^{\pm1}, \ldots, u_d^{\pm1}]$ and thus there is a non-zero integer $k \in \mathbb{Z}$ such that $k(u_j^l - 1)^N \in \mathfrak{p}$ for all j. Since $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, we conclude that $(u_j^l - 1)^N \in \mathfrak{p}$ and by primality $u_j^l - 1 \in \mathfrak{p}$ for all j.

Otherwise, we have $p > 0$. Since $p \in \mathfrak{p}$, we have that $\mathfrak{p}$ is equal to the pre-image of $\mathfrak{a}$ under the epimorphism $R_d \twoheadrightarrow F[u_1^{\pm1}, \ldots, u_d^{\pm1}]$. Thus $(u_j^l - 1)^N \in \mathfrak{a}$ immediately implies $(u_j^l - 1)^N \in \mathfrak{p}$ for all j and we may conclude using primality of $\mathfrak{p}$. $\square$

Example 3.15. We provide an example of an algebraic $\mathbb{Z}^d$-action (X, α) such that α is ergodic but $\alpha_{\mathbf{n}}$ is non-ergodic for every $\mathbf{n} \in \mathbb{Z}^d$.[5] This shows the necessity of the Noetherian condition for part (d) of Theorem 3.12.

Let $d \geq 2$ and set $N_{\mathbf{n}} = R_d/(u^{\mathbf{n}} - 1)$ for all $\mathbf{n} \in \mathbb{Z}^d$. Then, for each $\mathbf{n} \in \mathbb{Z}^d$, we have a decomposition $u^{\mathbf{n}} - 1 = \prod_{j=1}^{k_{\mathbf{n}}} p_{\mathbf{n},j}^{e_{\mathbf{n},j}}$ into irreducible elements $p_{\mathbf{n},j} \in R_d$. As previously discussed in Section 3.2, $\operatorname{Ass}(N_{\mathbf{n}}) = \{(p_{\mathbf{n},j}) : j = 1, \ldots, k_{\mathbf{n}}\}$. Since $d \geq 2$, the variety $\mathrm{V}((p_{\mathbf{n},j}))$ defined by a principal ideal is infinite (see Proposition C.10). Recalling that the set of l-th roots of unity in $\overline{\mathbb{F}}_p$ is finite for every $l \geq 1$, we see that Theorem 3.12 implies that $\alpha^{N_{\mathbf{n}}}$ is ergodic for every $\mathbf{n} \in \mathbb{Z}^d$.

Now define the R_d-module $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^d} N_{\mathbf{n}}$. Since $\operatorname{Ass}(M) = \bigcup_{\mathbf{n} \in \mathbb{Z}^d} \operatorname{Ass}(N_{\mathbf{n}})$, Theorem 3.12 implies that α^M is ergodic. Finally, we will show that $\alpha_{\mathbf{n}}^M$ is non-ergodic for every $\mathbf{n} \in \mathbb{Z}^d$ using Lemma 3.14. Indeed, given an $\mathbf{n} \in \mathbb{Z}^d$, we see that by construction any prime ideal associated to $N_{\mathbf{n}}$ contains $u^{\mathbf{n}} - 1$ and thus that $\alpha_{\mathbf{n}}^M$ is non-ergodic.

Theorem 3.16. *Let M be a countable R_d-module and $(X, \alpha) = (X^M, \alpha^M)$ as obtained from Proposition 3.2. Then the following are equivalent:*

- *(a) α is mixing.*
- *(b) $\alpha_{\mathbf{n}}$ is mixing for every non-zero element $\mathbf{n} \in \mathbb{Z}^d$.*
- *(c) $\alpha_{\mathbf{n}}$ is ergodic for every non-zero element $\mathbf{n} \in \mathbb{Z}^d$.*
- *(d) $\alpha^{R_d/\mathfrak{p}}$ is mixing for every prime ideal $\mathfrak{p}$ associated to M.*
- *(e) None of the prime ideals associated to M contains a polynomial of the form $u^{\mathbf{n}} - 1$ for any $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \{0\}$.*

Proof. The equivalence of (a), (b) and (c) is an immediate consequence of Theorem 2.14. Applying these equivalences and Lemma 3.14 to M and $R_d/\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ associated to M concludes the proof. $\square$

3.3.3 Expansiveness

We write $\mathbb{S}^d$ for $(\mathbb{S}^1)^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$.

Theorem 3.17. *Let M be a Noetherian R_d-module with associated primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$, and $(X, \alpha) = (X^M, \alpha^M)$ as obtained from Proposition 3.2. Then the following are equivalent:*

- *(a) α is expansive.*
- *(b) $\alpha^{R_d/\mathfrak{p}_i}$ is expansive for every $i = 1, \ldots, m$.*
- *(c) $\mathrm{V}_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d = \varnothing$ for every $i = 1, \ldots, m$.*

Remark 3.18. We note that the Noetherian condition in Theorem 3.17 is necessary in the sense that the dual module of any expansive algebraic $\mathbb{Z}^d$-action must be Noetherian. This follows from combining Theorem 2.8, Corollary 2.7, and Proposition 3.10.

Lemma 3.19. *Let $\mathfrak{a} \subseteq R_d$ be an ideal and suppose that $(X^{R_d/\mathfrak{a}}, \alpha^{R_d/\mathfrak{a}})$, as obtained from Proposition 3.2, is not expansive. Then we have $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d \neq \varnothing$.*

Before we prove the lemma, we wish to discuss the idea of the argument. Our goal is to construct a point in $\mathbb{S}^d$ that also lies in the "spectrum" $\mathrm{V}_{\mathbb{C}}(\mathfrak{a})$ of $\alpha^{R_d/\mathfrak{a}}$. By Example 3.4 and Proposition 3.3, we see that $\alpha^{R_d/\mathfrak{a}}$ is given by a subshift of $\mathbb{T}^{\mathbb{Z}^d}$. In an attempt to relate $\mathrm{V}_{\mathbb{C}}(\mathfrak{a})$ to the "joint spectrum" of the shift operators, we will "linearize" our ambient space $\mathbb{T}^{\mathbb{Z}^d}$ to the Banach space $\ell^\infty(\mathbb{Z}^d)$ of all bounded complex-valued functions on $\mathbb{Z}^d$ in

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[5] We invite the reader to think about why this example will necessarily satisfy $d \geq 2$.

the supremum norm. We will then consider the commutative Banach algebra generated by the isometries $\{U_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ of $\ell^\infty(\mathbb{Z}^d)$ that are defined by $(U_{\mathbf{n}} z)_{\mathbf{m}} = z_{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$ and $z \in \ell^\infty(\mathbb{Z}^d)$ (which are the linearizations of the shift operators $\alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}$). The assumption that $\alpha^{R_d/\mathfrak{a}}$ is not expansive will imply that a special subspace S of $\ell^\infty(\mathbb{Z}^d)$ is non-trivial. Finally, we will use the Gelfand transform (see Appendix D) to analyze the joint spectrum of the restrictions of the operators $U_{\mathbf{n}}$ to S to obtain our desired point in $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$.

Proof of Lemma 3.19. If $\mathfrak{a} = (0)$, then $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) = (\mathbb{C}^\times)^d$ and in particular $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d \neq \varnothing$. So we assume that $\mathfrak{a} \neq (0)$. Since R_d is Noetherian, $\mathfrak{a}$ is generated by finitely many elements, say $f_1, \ldots, f_r$. Using Example 3.4 and Proposition 3.3, we may identify $(X^{R_d/\mathfrak{a}}, \alpha^{R_d/\mathfrak{a}})$ as

$$X^{R_d/\mathfrak{a}} = \left\{ (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) x_{\mathbf{n}+\mathbf{m}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ and } j \in \{1, \ldots, r\} \right\}$$

with $\alpha^{R_d/\mathfrak{a}}$ given by the shift-action.

Consider the linear subspace

$$S = \left\{ (z_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^d) : \sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) z_{\mathbf{n}+\mathbf{m}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ and } j \in \{1, \ldots, r\} \right\}$$

of the Banach space $\ell^\infty(\mathbb{Z}^d)$ of bounded, complex-valued functions on $\mathbb{Z}^d$ in the supremum norm. Let $U_{\mathbf{n}}$ denote the invertible, isometric operator on $\ell^\infty(\mathbb{Z}^d)$ given by $(U_{\mathbf{n}} z)_{\mathbf{m}} = z_{\mathbf{n}+\mathbf{m}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$ and all $z = (z_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d)$. We may then rewrite the condition in the definition of S as $\left(\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) U_{\mathbf{n}}\right) z = 0$ for all $j = 1, \ldots, r$. This shows that S is an intersection of kernels of continuous linear operators on $\ell^\infty(\mathbb{Z}^d)$ and thus itself a Banach space. Let $V_{\mathbf{n}}$ denote the restriction of $U_{\mathbf{n}}$ to S for all $\mathbf{n} \in \mathbb{Z}^d$, and consider the commutative Banach algebra $\mathcal{A}$ generated by the invertible, isometric operators $\{V_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$.

Next, we define $\|f\| = \sum_{\mathbf{n}\in\mathbb{Z}^d} |c_f(\mathbf{n})|$ for any polynomial $f(u) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in R_d$, and set $\varepsilon = \left(3 \sum_{j=1}^r \|f_j\|\right)^{-1} > 0$. Consider the neighborhood of the identity

$$N = \left\{ (x_{\mathbf{n}}) \in X^{R_d/\mathfrak{a}} : d_{\mathbb{T}}(x_{\mathbf{0}}, 0) < \varepsilon \right\}.$$

Since we assumed $\alpha^{R_d/\mathfrak{a}}$ to be non-expansive, the intersection $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}(N)$ is non-trivial. So there exists a non-zero point $(x_{\mathbf{n}}) \in X^{R_d/\mathfrak{a}}$ such that $d_{\mathbb{T}}(x_{\mathbf{n}}, 0) < \varepsilon$ for all $\mathbf{n} \in \mathbb{Z}^d$. Let $(y_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^d)$ denote the unique lift of $(x_{\mathbf{n}})$ to $\mathbb{R}^{\mathbb{Z}^d}$ (via the canonical covering map $\mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$) that satisfies $|y_{\mathbf{n}}| < \varepsilon$ for all $\mathbf{n} \in \mathbb{Z}^d$. Uniqueness is guaranteed by the fact that at least one of the f_j's is non-zero and thus $\varepsilon < 1/2$. Since $x_{\mathbf{n}} = y_{\mathbf{n}} \bmod 1$ for all $\mathbf{n} \in \mathbb{Z}^d$, we have that $\left(\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) U_{\mathbf{n}}\right) y$ has all coordinates in $\mathbb{Z}$. By our choice of ε, we further deduce that

$$\left\| \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) U_{\mathbf{n}} \right) y \right\| \leq \sum_{\mathbf{n}\in\mathbb{Z}^d} |c_{f_j}(\mathbf{n})| \|U_{\mathbf{n}}\| \|y\| \leq \|f_j\| \varepsilon < 1,$$

for all $j = 1, \ldots, r$. Since x is non-zero, its lift y is non-zero. This shows that $y \in S$ and, in particular, that $S \neq \{0\}$. Consequently, the identity operator on S is not the zero operator, ensuring that the commutative Banach algebra $\mathcal{A}$ generated by the restrictions $V_{\mathbf{n}}$ is a unital Banach algebra ($\mathrm{id}_S \neq 0$ in $\mathcal{A}$).

Now consider the Gelfand transform

$$\begin{aligned} \mathcal{A} &\to C(\Delta_{\mathcal{A}}) \\ A &\mapsto \hat{A}, \end{aligned}$$

where $\Delta_{\mathcal{A}}$ denotes the Gelfand spectrum of $\mathcal{A}$. By Theorem D.9, the Gelfand transform is a norm-non-increasing algebra homomorphism. Thus, we have $|\hat{V}_{\mathbf{n}}(\omega)| \leq \|\hat{V}_{\mathbf{n}}\| \leq \|V_{\mathbf{n}}\| = 1$ for all $\mathbf{n} \in \mathbb{Z}^d$ and $\omega \in \Delta_{\mathcal{A}}$. Since the Gelfand spectrum consists of non-trivial algebra homomorphisms, we further observe that $\hat{\mathrm{id}}_S(\omega) = \omega(\mathrm{id}_S) = 1$ for all $\omega \in \Delta_{\mathcal{A}}$, which implies that $1 = |\hat{\mathrm{id}}_S(\omega)| \leq |\hat{V}_{\mathbf{n}}(\omega)||\hat{V}_{-\mathbf{n}}(\omega)|$ for all $\mathbf{n} \in \mathbb{Z}^d$ and all $\omega \in \Delta_{\mathcal{A}}$. Putting these two observations together, we obtain $|\hat{V}_{\mathbf{n}}(\omega)| = 1$ for all $\omega \in \Delta_{\mathcal{A}}$ and $\mathbf{n} \in \mathbb{Z}^d$.

Fix some $\omega \in \Delta_{\mathcal{A}}$ and define $\lambda_k = \hat{V}_{\mathbf{e}^{(k)}}(\omega)$ for all $k = 1, \dots, d$, where $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(d)}$ denotes the standard $\mathbb{Z}$-basis of $\mathbb{Z}^d$. By our previous observation, we have that the point $\lambda = (\lambda_1, \dots, \lambda_d)$ lies in $\mathbb{S}^d$. We observe that

$$\hat{V}_{\mathbf{n}}(\omega) = \widehat{(V_{\mathbf{e}^{(1)}}^{n_1} \circ \cdots \circ V_{\mathbf{e}^{(d)}}^{n_d})}(\omega) = \hat{V}_{\mathbf{e}^{(1)}}^{n_1}(\omega) \cdots \hat{V}_{\mathbf{e}^{(d)}}^{n_d}(\omega) = \lambda^{\mathbf{n}}$$

for any $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$. Furthermore, we recall that by definition of S, we have $\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n}) V_{\mathbf{n}} = 0$ for all $j = 1, \dots, r$. Combining these two, we obtain

$$0 = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n}) \hat{V}_{\mathbf{n}}(\omega) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n}) \lambda^{\mathbf{n}} = f_j(\lambda),$$

for all $j = 1, \dots, r$. This shows that $\lambda \in \mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$, concluding the proof. □

The next lemma is the converse of Lemma 3.19 and thus establishes the equivalence that $\alpha^{R_d/\mathfrak{a}}$ is expansive if and only if $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d = \varnothing$. The main purpose of this lemma is, however, to present a simpler case of the argument that we will later use to prove the implication (a) $\implies$ (c) of Theorem 3.17.

Lemma 3.20. *Let $\mathfrak{a} \subseteq R_d$ be an ideal such that $\mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d \neq \varnothing$. Then $(X^{R_d/\mathfrak{a}}, \alpha^{R_d/\mathfrak{a}})$, as obtained from Proposition 3.2, is not expansive.*

Proof. Let $\lambda \in \mathrm{V}_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$ and consider the ring homomorphism

$$\begin{aligned} R_d &\to \mathbb{C} \\ f &\mapsto f(\lambda), \end{aligned}$$

making $(\mathbb{C}, +)$ into an R_d-module. Since $\lambda \in \mathrm{V}_{\mathbb{C}}(\mathfrak{a})$, the kernel of the above ring homomorphism contains $\mathfrak{a}$ and thus descends to an R_d-module homomorphism $\eta : R_d/\mathfrak{a} \to \mathbb{C}$. If we denote by $\hat{\theta}$ and $\hat{\alpha}^{R_d/\mathfrak{a}}$ the $\mathbb{Z}^d$-actions given by multiplication with monomials of R_d on $\mathbb{C}$ and $R_d/\mathfrak{a}$ respectively, then we have

$$\eta \circ \hat{\alpha}_{\mathbf{n}}^{R_d/\mathfrak{a}} = \hat{\theta}_{\mathbf{n}} \circ \eta \tag{3.2}$$

for all $\mathbf{n} \in \mathbb{Z}^d$, because η is R_d-linear.

Let W denote the closure of the image $\eta(R_d/\mathfrak{a})$ in $\mathbb{C}$ with respect to the standard topology. Notice that as a closed subgroup of $(\mathbb{C}, +)$, W is a locally compact abelian group. As usual, we equip $R_d/\mathfrak{a}$ with the discrete topology, making $\eta : R_d/\mathfrak{a} \to W$ a continuous homomorphism of locally compact abelian groups. Therefore, there is a continuous dual homomorphism $\hat{\eta} : \widehat{W} \to X^{R_d/\mathfrak{a}}$ (see Proposition A.11). Let $\chi \in \ker \hat{\eta}$. Then we have $\chi(\eta(f + \mathfrak{a})) = 1$ for all $f + \mathfrak{a} \in R_d/\mathfrak{a}$, and continuity of χ implies that $\chi(w) = 1$ for all w

in the closure W of $\eta(R_d/\mathfrak{a})$. This shows that $\hat{\eta}$ is injective. Furthermore, using Eq. (3.2), continuity of η, and density of $\eta(R_d/\mathfrak{a})$ in W, we see that W is $\hat{\theta}$-invariant. It is readily checked that Eq. (3.2) implies that

$$\hat{\eta} \circ \theta_{\mathbf{n}} = \alpha_{\mathbf{n}}^{R_d/\mathfrak{a}} \circ \hat{\eta}, \tag{3.3}$$

for all $\mathbf{n} \in \mathbb{Z}^d$, where $\theta_{\mathbf{n}}$ and $\alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}$ denote the dual automorphisms induced by $\hat{\theta}_{\mathbf{n}}$ and $\hat{\alpha}_{\mathbf{n}}^{R_d/\mathfrak{a}}$ respectively. We note here that by evaluating constant polynomials at λ, we see that W contains the integers $\mathbb{Z}$. This implies that W is non-compact and thus $\widehat{W}$ is not discrete (see Proposition A.9).

Recall that $\widehat{W} \subseteq C(W, \mathbb{S}^1)$ is equipped with the compact-open topology. Consider the cover of W given by the compact subsets $D_m = \{w \in W : |w| \leq m\}$ for all $m \geq 1$. For any $\chi, \mu \in \widehat{W}$ and any integer $m \geq 1$, we set $d_m(\chi, \mu) = \sup_{w \in D_m} |\chi(w) - \mu(w)|$. Then

$$d(\chi, \mu) = \sum_{m \geq 1} 2^{-m} \frac{d_m(\chi, \mu)}{1 + d_m(\chi, \mu)},$$

defines a compatible metric on $\widehat{W}$. Recalling that $\hat{\theta}_{\mathbf{n}}(w) = \lambda^{\mathbf{n}} w$ and $\lambda \in \mathbb{S}^d$, we see that $\hat{\theta}_{\mathbf{n}}(D_m) = D_m$ holds for all $m \geq 1$ and all $\mathbf{n} \in \mathbb{Z}^d$. Thus $d(\theta_{\mathbf{n}}(\chi), \theta_{\mathbf{n}}(\mu)) = d(\chi, \mu)$ for all $\chi, \mu \in \widehat{W}$ and $\mathbf{n} \in \mathbb{Z}^d$. This shows in particular that the family $\{\theta_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ of automorphisms is equicontinuous on $\widehat{W}$.

Let N be any open neighborhood of the identity in $X^{R_d/\mathfrak{a}}$. By continuity of $\hat{\eta}$, the pre-image $\hat{\eta}^{-1}(N)$ is an open neighborhood of the identity in $\widehat{W}$. By equicontinuity of the automorphisms $\theta_{\mathbf{n}}$, we may find another open neighborhood of the identity U in $\widehat{W}$ such that $\theta_{\mathbf{n}}(U) \subseteq \hat{\eta}^{-1}(N)$ for all $\mathbf{n} \in \mathbb{Z}^d$. Since $\widehat{W}$ is not discrete and U is open, there exists a non-identity element $u \in U$. We set $x = \hat{\eta}(u) \in N$ and note that by injectivity of $\hat{\eta}$, x is non-trivial in $X^{R_d/\mathfrak{a}}$. Now using Eq. (3.3) we see that

$$\alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}(x) = \alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}(\hat{\eta}(u)) = \hat{\eta}(\theta_{\mathbf{n}}(u)) \in \hat{\eta}(\hat{\eta}^{-1}(N)) \subseteq N,$$

for all $\mathbf{n} \in \mathbb{Z}^d$. This shows that $\bigcap_{\mathbf{n} \in \mathbb{Z}^d} \alpha_{\mathbf{n}}^{R_d/\mathfrak{a}}(N) \neq \{1_{X^{R_d/\mathfrak{a}}}\}$ and thus that N is not an expansive neighborhood. Since N was arbitrary, this concludes the proof. □

Proof of Theorem 3.17. (c) $\implies$ (a): Let $\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a prime filtration of M (see Proposition C.13). Then, for all $j \in \{1, \ldots, n\}$, we have that $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$ for some prime ideal $\mathfrak{q}_j$ such that $\mathfrak{p}_i \subseteq \mathfrak{q}_j$ for some $i \in \{1, \ldots, m\}$. Setting $X_j = M_j^{\perp}$ for all $j \in \{0, \ldots, n\}$, Proposition 3.3 shows that we have the chain of closed, α-invariant subgroups $\{0\} = X_n \subseteq \cdots \subseteq X_1 \subseteq X_0 = X$. Let $j \in \{1, \ldots, n\}$. We observe that $X_{j-1} \cong \widehat{M/M_{j-1}}$ and that $\widehat{X_{j-1}/X_j}$ corresponds to taking the annihilator of X_j as a subgroup of X_{j-1} (Proposition A.7). It is then readily checked that $\widehat{X_{j-1}/X_j} \cong M_j/M_{j-1}$ and hence $(X_{j-1}/X_j, \alpha^{X_{j-1}/X_j})$ is conjugate to the action dual to $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$.

Since $\mathfrak{p}_i \subseteq \mathfrak{q}_j$ for some $i \in \{1, \ldots, m\}$, we have $\mathrm{V}_{\mathbb{C}}(\mathfrak{q}_j) \subseteq \mathrm{V}_{\mathbb{C}}(\mathfrak{p}_i)$ and thus Lemma 3.19 implies that α^{X_{j-1}/X_j} is expansive. Since α^{X_n} is trivially expansive, we can, starting with $j = n$, iteratively apply Lemma 1.29 to α^{X_{j-1}/X_j} and α^{X_j} and thus conclude that $\alpha^{X_0} = \alpha$ is expansive.

(a) $\implies$ (c): The following argument adapts the proof of Lemma 3.20, with a few additional complications. Suppose there exists a $\lambda \in \mathrm{V}_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d$ for some $i \in \{1, \ldots, m\}$ and fix this i. Let $\{0\} = M_1 \cap \cdots \cap M_m$ be a minimal primary decomposition of $\{0\}$ in M, indexed such that M_j is $\mathfrak{p}_j$-primary for all $j \in \{1, \ldots, m\}$ (see Theorem C.18). We set $M' = M/M_i$. Our goal is to construct an R_d-module homomorphism $\eta : M' \to \mathbb{C}$, where the module structure on $\mathbb{C}$ is induced by the ring homomorphism $\mathrm{ev}_\lambda : R_d \to \mathbb{C}$ given by $f \mapsto f(\lambda)$. From that point on, the argument proceeds analogously to the proof of Lemma 3.20: We will use the fact that λ lies on $\mathbb{S}^d$ to see that the dual homomorphism $\hat{\eta}$ gives rise to a subgroup of X on which α is equicontinuous and hence non-expansive.

Let $\mathfrak{a} = \ker(\mathrm{ev}_\lambda)$. We claim that there exists an element $g_1 \in M'$ such that for all $f \in R_d$, $fg_1 \in \mathfrak{a}M'$ implies $f \in \mathfrak{a}$. Suppose for a contradiction that no such element exists. Then, since M is Noetherian, M' is finitely generated, say, by $m_1, \dots, m_k$. Then, by our assumption, there exist elements $s_i \in R_d \setminus \mathfrak{a}$ such that $s_i m_i \in \mathfrak{a}M'$ for all $i \in \{1, \dots, k\}$. Setting $s = s_1 \cdots s_k$, we have $sM' \subseteq \mathfrak{a}M'$. We will now show that $s \in \mathfrak{a}$, which will be our desired contradiction because $\mathfrak{a}$ is a prime ideal and $s_i \notin \mathfrak{a}$ for all $i \in \{1, \dots, k\}$. Indeed, since $sM' \subseteq \mathfrak{a}M'$, the Cayley–Hamilton theorem (Lemma C.4) implies that there exist $c_0, \dots, c_{n-1} \in \mathfrak{a}$ such that multiplication by $t = s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0$ is the zero endomorphism on M', i.e., $t \in \mathrm{Ann}(M')$. Recalling $\mathrm{Ass}(M') = \{\mathfrak{p}_i\}$ and $\lambda \in \mathrm{V}_{\mathbb{C}}(\mathfrak{p}_i)$, we see that $\mathrm{Ann}(M') \subseteq \mathfrak{p}_i \subseteq \mathfrak{a}$. Thus, we obtain $s^n \in \mathfrak{a}$ and, by primality, $s \in \mathfrak{a}$.

We have thus established the existence of our desired element $g_1 \in M'$. We extend g_1 to a generating set $g_1, \dots, g_k$ of M' and define the surjective R_d-module homomorphism $\zeta : R_d^k \to M'$ by $(f_1, \dots, f_k) \mapsto f_1 g_1 + \cdots + f_k g_k$.

We define the submodules $L = \ker \zeta + \mathfrak{a}^{\oplus k}$ and $N = \{(f, 0, \dots, 0) \in R_d^k : f \in R_d\}$, and the map

$$\begin{aligned} \phi : L + N &\to \mathbb{C} \\ a + b &\mapsto \mathrm{ev}_\lambda(f), \end{aligned}$$

where $a \in L$ and $b = (f, 0, \dots, 0) \in N$. We claim that $\phi(L \cap N) = 0$, which implies that ϕ is a well-defined map. Indeed, if $x \in L \cap N$, then by the definition of L we may write $x = (v_1, \dots, v_k) + (w_1, \dots, w_k)$, for some $(v_1, \dots, v_k) \in \ker \zeta$ and some $(w_1, \dots, w_k) \in \mathfrak{a}^{\oplus k}$. Since $x \in N$, we have $v_j = -w_j$ for all $j = 2, \dots, k$. Using $(v_1, \dots, v_k) \in \ker \zeta$ this yields $v_1 g_1 = \sum_{j=2}^k w_j g_j$. Since $w_j \in \mathfrak{a}$ for all $j \in \{2, \dots, k\}$, we obtain $v_1 g_1 \in \mathfrak{a}M'$. By our choice of g_1, this implies $v_1 \in \mathfrak{a}$. This finally yields that $v_1 + w_1 \in \mathfrak{a}$ and thus $\phi(x) = 0$. To see that ϕ is an R_d-module homomorphism, let $\mathbf{n} \in \mathbb{Z}^d$ and observe that

$$\phi(u^{\mathbf{n}} \cdot (a + b)) = \mathrm{ev}_\lambda(u^{\mathbf{n}} f) = \lambda^{\mathbf{n}} f(\lambda) = u^{\mathbf{n}} \cdot \mathrm{ev}_\lambda(f) = u^{\mathbf{n}} \cdot \phi(a + b),$$

where $a \in L$ and $b = (f, 0, \dots, 0) \in N$.

We claim that there is an extension of ϕ to R_d^k. Indeed, consider the set $\mathcal{M}$ of all tuples (ψ, P) such that P is a submodule of R_d^k containing $L + N$ and $\psi : P \to \mathbb{C}$ is an R_d-module homomorphism with $\psi|_{L+N} = \phi$. The set $\mathcal{M}$ has a natural partial ordering given by $(\psi, P) \leq (\psi', P')$ if and only if $P \subseteq P'$ and $\psi = \psi'|_P$. Furthermore, $\mathcal{M}$ contains $(\phi, L + N)$ and is thus non-empty. Assume we have a chain $\{(\psi_j, P_j)\}_{j \in J}$ in $\mathcal{M}$. Setting $P = \bigcup_{j \in J} P_j$ and defining $\psi : P \to \mathbb{C}$ by $\psi(x) = \psi_j(x)$ whenever $x \in P_j \subseteq P$, we see that the chain has an upper bound in $\mathcal{M}$. So we may apply Zorn's lemma to obtain a maximal element (φ, Q) of $\mathcal{M}$.

For a contradiction, suppose there exists a $b \in R_d^k \setminus Q$. If $\varphi(fb) = 0$ for all $fb \in R_d b \cap Q$, then we set $\rho = 0$. Otherwise, there exists an $fb \in R_d b \cap Q$ such that $\varphi(fb) \neq 0$. In this case $f(\lambda) \neq 0$, since otherwise we would have $f \in \mathfrak{a}$ and thus $fb \in \mathfrak{a}^{\oplus k} \subseteq L$, giving $\varphi(fb) = \phi(fb + 0) = 0$. So we may set $\rho = \varphi(fb)/f(\lambda)$.

We define the R_d-linear map

$$\begin{aligned} \tilde{\varphi} : R_d b + Q &\to \mathbb{C} \\ gb + q &\mapsto g(\lambda)\rho + \varphi(q). \end{aligned}$$

To see that $\tilde{\varphi}$ is well-defined, we must show that if $gb \in R_d b \cap Q$, then $\varphi(gb) = g(\lambda)\rho$. Indeed, let $gb \in R_d b \cap Q$ and observe that also $fgb \in R_d b \cap Q$. Since φ is an R_d-module homomorphism, we can compute $\varphi(fgb)$ in two ways:

$$\varphi(fgb) = f(\lambda)\varphi(gb) \quad \text{and} \quad \varphi(fgb) = g(\lambda)\varphi(fb) = g(\lambda)f(\lambda)\rho.$$

Equating these and using the fact that $f(\lambda) \neq 0$, we obtain $\varphi(gb) = g(\lambda)\rho$, ensuring that $\tilde{\varphi}$ is well-defined. Thus, we have found a proper extension of $\varphi : Q \to \mathbb{C}$, yielding our desired contradiction. Therefore, it must be that $Q = R_d^k$ and thus φ is an extension of ϕ to R_d^k.

Let $x \in \ker\zeta \subseteq L \subseteq R_d^k$. Then, by construction, we have $\varphi(x) = \phi(x) = 0$. This shows that $\ker\zeta \subseteq \ker\varphi$ and thus φ descends to an R_d-module homomorphism $\eta : M' \to \mathbb{C}$. Let W denote the closure of the image $\eta(M')$ in $\mathbb{C}$ with respect to the standard topology. We had already seen that $L \cap N \subseteq \{(f, 0, \dots, 0) \in R_d^k : f \in \mathfrak{a}\}$. Thus, the image of $\{(n, 0, \dots, 0) \in R_d^k : n \in \mathbb{Z}\}/\ker\zeta \subseteq M'$ under η is equal to $\mathbb{Z} \subseteq \mathbb{C}$. This shows in particular that W is not compact. Consider the dual homomorphism $\hat{\eta} : \widehat{W} \to \widehat{M'}$. Note that $\widehat{M'} = \widehat{M/M_i} \cong M_i^{\perp}$. Thus, by Proposition 3.3, $\widehat{M'}$ corresponds to a closed, α-invariant subgroup of X. We therefore have an injective, continuous group homomorphism $\hat{\eta} : \widehat{W} \to X$ such that $\hat{\eta} \circ \theta_{\mathbf{n}} = \alpha_{\mathbf{n}} \circ \hat{\eta}$ for all $\mathbf{n} \in \mathbb{Z}^d$, where $\theta_{\mathbf{n}}$ is the automorphism dual to multiplication by $u^{\mathbf{n}}$ on W.

From here on, the argument proceeds almost verbatim to the one presented in the proof of Lemma 3.20, showing that α is non-expansive.

(b) $\iff$ (c): This follows immediately from Lemmas 3.19 and 3.20 or by applying the equivalence (a) $\iff$ (c) to the modules $R_d/\mathfrak{p}_i$. ☐

3.4 Applications and Examples

In Chapter 1, we analyzed the dynamical behavior of our four running examples using a variety of ad hoc methods. By applying the algebraic dictionary developed in this chapter, we will show how these results can be quickly recovered with the unified approach of examining the prime ideals associated to their dual modules.

Subsequently, we will demonstrate how this algebraic dictionary may be used to produce a wealth of examples satisfying specific dynamical properties.

Example 3.21 (Arnold's Cat Map VI)**.** We revisit Arnold's Cat Map for the last time. Recall that it is the toral automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Combining Example 1.37 and Theorem 3.11, we immediately obtain that the set of T_A-periodic points is dense in $\mathbb{T}^2$.

Next, recall that the characteristic polynomial of A is given by $f_A(u) = u^2 - 3u + 1$. The companion matrix of f_A is given by $C = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 3 \end{smallmatrix}\right)$, so by setting $P = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, we see that $A = PCP^{-1}$. This shows that we may apply our derivation from Example 3.7 to obtain that the dual module of T_A is isomorphic to $R_1/(f_A)$. Since f_A is irreducible in R_1, we have that $\operatorname{Ass}(R_1/(f_A)) = \{(f_A)\}$.

Since f_A is not a cyclotomic polynomial, Theorem 3.16 implies that T_A is mixing, and thus ergodic and topologically transitive by Lemma 1.20 and Corollary 2.16. Furthermore, because the roots of $f_A(u)$, which are $\frac{3\pm\sqrt{5}}{2}$, do not lie on the unit circle, applying Theorem 3.17 yields that T_A is expansive.

Example 3.22 (Toral Rotation V)**.** Recall that the Toral Rotation is the toral automorphism $T_R : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the matrix $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The characteristic polynomial is $f_R(u) = u^2 + 1$ and so the companion matrix is given by $C = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We see that R and C

are conjugate over $\mathbb{Z}$ via the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, we may again conclude using Example 3.7 that the dual module of T_R is isomorphic to $R_1/(f_R)$. Since f_R is irreducible in R_1, we have that $\operatorname{Ass}(R_1/(f_R)) = \{(f_R)\}$. Noticing that $f_R \cdot (u^2 - 1) = u^4 - 1$, we see that the ideal (f_R) contains the set $\{u^{4k} - 1 : k \in \mathbb{Z}\}$. Thus, Theorem 3.12 implies that T_R is non-ergodic. Furthermore, since $\pm i \in \mathrm{V}_{\mathbb{C}}((f_R)) \cap \mathbb{S}^1$, we see that T_R is non-expansive using Theorem 3.17.

Example 3.23 (Square Shift VII)**.** Recalling Example 3.8, we see that the dual module of the Square Shift $\sigma : \mathbb{Z}^2 \to \operatorname{Aut}(X)$ is isomorphic to $R_2/(2, f)$, where $f(u_1, u_2) = (u_1+1)(u_2+1)$.

Recalling that R_2 is Noetherian and using Theorem 3.11, we immediately obtain that the subgroup of σ-periodic points is dense in X.

Next, we wish to determine the prime ideals associated to $R_2/(2, f)$. First, consider the sequence of isomorphisms

$$R_2/(2, f) \cong \frac{R_2/(2)}{(2, f)/(2)} \cong \mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]/(\bar{f}),$$

where $\bar{f}$ is the projection of f under the surjection $\pi : R_2 \to \mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]$ given by reducing the coefficients modulo 2. Hence, $\bar{f}(u_1, u_2) = (u_1 + 1)(u_2 + 1)$. Notice that both $u_1 + 1$ and u_2+1 are irreducible in $\mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]$ and hence $(\bar{f}) = (u_1+1) \cap (u_2+1)$ is a minimal primary decomposition of $(\bar{f})$. This shows that the prime ideals associated to $\mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]/(\bar{f})$ are exactly $(u_1 + 1)$ and $(u_2 + 1)$. Thus, by taking the pre-images under π, we see that

$$\operatorname{Ass}(R_2/(2, f)) = \{(2, u_1 + 1), (2, u_2 + 1)\}.$$

Since $(2, u_1 + 1)$ contains the polynomial $u_1^2 - 1$, Lemma 3.14 implies that $\sigma_{(1,0)}$ is non-ergodic. We further observe that neither $(2, u_1 + 1)$ nor $(2, u_2 + 1)$ contains an element of the form $u_1^l u_2^l - 1$ for any integer $l \geq 1$. This can, for example, be checked by considering the ring homomorphism $R_d \to \mathbb{Z}[u_2^{\pm 1}]$ given by $h(u_1, u_2) \mapsto h(-1, u_2)$. This implies, again via Lemma 3.14, that $\sigma_{(1,1)}$ is ergodic. Using Theorems 3.12 and 3.16, we conclude that σ is ergodic but not mixing.

Finally, since both associated prime ideals contain a constant, the induced complex varieties are empty. Thus, Theorem 3.17 implies that σ is expansive.

Example 3.24 (Times Two Map V)**.** We return to the Times Two Map $T_2 : \mathbb{T} \to \mathbb{T}$, given by $x \mapsto 2x$, and finally resolve the question of how it is “related to” an algebraic $\mathbb{Z}$-action.

Recall that we may view the Times Two Map as a monoid action of $\mathbb{Z}_{\geq 0}$ on $\mathbb{T}$ by continuous surjective endomorphisms. For that, we define $\beta : \mathbb{Z}_{\geq 0} \to \operatorname{End}(\mathbb{T})$ by $k \mapsto T_2^k$. We will denote this system by $(\mathbb{T}, \beta)$. We set

$$\mathbb{S}_2 = \left\{ (x_k)_{k \in \mathbb{Z}_{\geq 0}} \in \mathbb{T}^{\mathbb{Z}_{\geq 0}} : \beta_{k'}(x_{k'+k}) = x_k \text{ for all } k, k' \in \mathbb{Z}_{\geq 0} \right\},$$

which may also be viewed as the inverse limit of the system $(\mathbb{T}, \beta)$ and is known as the 2-adic solenoid. Further defining

$$\alpha_n(x_0, x_1, x_2, \dots) = \begin{cases} (\beta_n(x_0), \beta_n(x_1), \beta_n(x_2), \dots) & n > 0 \\ (x_{-n}, x_{-n+1}, x_{-n+2}, \dots) & n \leq 0, \end{cases}$$

for $n \in \mathbb{Z}$, we obtain an algebraic $\mathbb{Z}$-action $(\mathbb{S}_2, \alpha)$. Notice that by the definition of $\mathbb{S}_2$, for $k \geq n$, we have $\beta_n(x_k) = x_{k-n}$. Thus, for $n > 0$, α_n acts essentially as a right shift that creates new coordinates at the front by applying β, confirming it is the inverse of the left shift defined for $n \leq 0$.

Let $\pi : \mathbb{S}_2 \to \mathbb{T}$ denote the canonical projection to the 0th coordinate given by $(x_k)_{k \in \mathbb{Z}_{\geq 0}} \mapsto x_0$. Then, π is a continuous surjective group homomorphism and the diagram

$$\begin{tikzcd}
\mathbb{S}_2 \arrow[r, "\pi"] \arrow[d, "\alpha_k"'] & \mathbb{T} \arrow[d, "\beta_k"] \\
\mathbb{S}_2 \arrow[r, "\pi"'] & \mathbb{T}
\end{tikzcd}$$

commutes for every $k \in \mathbb{Z}_{\geq 0}$. In this sense, the Times Two Map is a *factor system* of the algebraic action $(\mathbb{S}_2, \alpha)$, which is the "relation" we have been referring to. The construction of $(\mathbb{S}_2, \alpha)$ is known as the *natural extension* and was first introduced by Rokhlin [Roh61] in the context of measurable dynamics.[6]

Next, we wish to determine the dual module of $(\mathbb{S}_2, \alpha)$ in order to apply our dictionary and analyze its dynamical behavior.

The 2-adic solenoid $\mathbb{S}_2$ is equipped with the subspace topology inherited from the product space $\mathbb{T}^{\mathbb{Z}_{\geq 0}}$. Let $(x_k) \in \mathbb{S}_2$ be a point, $K \geq 0$ an integer, and $\varepsilon > 0$ small. We notice that by the defining condition of $\mathbb{S}_2$, requiring $d_{\mathbb{T}}(x_K, 0) < \varepsilon$ also sets constraints on x_k for all $k \leq K$. This shows that the sets

$$U_{K,\varepsilon} = \{(x_k) \in \mathbb{S}_2 : d_{\mathbb{T}}(x_K, 0) < \varepsilon\},$$

where K ranges in $\mathbb{Z}_{\geq 0}$ and ε in $\mathbb{R}_{>0}$, form a fundamental system of open neighborhoods of $0 \in \mathbb{S}_2$.

Since $\mathbb{S}^1$ satisfies the no-small-subgroups property, there exists a neighborhood V of $1 \in \mathbb{S}^1$ such that V does not contain any non-trivial subgroup of $\mathbb{S}^1$. Let $\chi : \mathbb{S}_2 \to \mathbb{S}^1$ be a character of $\mathbb{S}_2$ and consider the pre-image $\chi^{-1}(V)$. By the above, there exists an integer $K \geq 0$ and an $\varepsilon > 0$ such that $U_{K,\varepsilon} \subseteq \chi^{-1}(V)$. Let $\pi_K : \mathbb{S}_2 \to \mathbb{T}$ denote the canonical projection to the K-th coordinate and notice that $\ker(\pi_K)$ is a closed subgroup of $\mathbb{S}_2$ that is contained in $U_{K,\varepsilon}$. Hence, $\chi(\ker(\pi_K))$ is a subgroup of $\mathbb{S}^1$ contained in V, which means it must be the trivial subgroup. This shows that χ factors through π_K. That is, there exists a character $\tilde{\chi} : \mathbb{T} \to \mathbb{S}^1$ such that $\chi = \tilde{\chi} \circ \pi_K$.

We have thus established that any character χ of $\mathbb{S}_2$ may be represented as

$$\chi(x) = \chi_{K,a}(x) = \exp(2\pi i a x_K)$$

for some $a \in \mathbb{Z}$ and $K \in \mathbb{Z}_{\geq 0}$, where $x = (x_k) \in \mathbb{S}_2$. Because $2x_K = x_{K-1}$ in $\mathbb{T}$, this representation is not unique. We have the equivalence $\chi_{K,2a} = \chi_{K-1,a}$, which leads us to claim that

$$\begin{aligned} \Phi : \widehat{\mathbb{S}_2} &\to \mathbb{Z}[1/2] \\ \chi_{K,a} &\mapsto \frac{a}{2^K} \end{aligned}$$

defines an isomorphism. We leave the verification of this claim to the reader.

Let $x = (x_k) \in \mathbb{S}_2$, $\frac{a}{2^K} \in \mathbb{Z}[1/2]$ and $n \in \mathbb{Z}$. We compute

$$\begin{aligned} \langle x, \hat{\alpha}_n(\tfrac{a}{2^K}) \rangle &= \langle \alpha_n(x), \tfrac{a}{2^K} \rangle \\ &= \begin{cases} \exp(2\pi i a x_{K-n}) & n \leq 0 \\ \exp(2\pi i a 2^n x_K) & n > 0 \end{cases} \\ &= \langle x, \tfrac{a}{2^{K-n}} \rangle, \end{aligned}$$

---

[6] We refer the reader to [BBD25] for a more extensive discussion on extending actions of *embeddable* monoids to group actions. We also refer the reader to [Lac95] for a discussion of how the probabilistic properties of ergodicity and mixing translate between the base action and the natural extension.

which shows that scalar multiplication by u_1^n in $\mathbb{Z}[1/2]$ corresponds to multiplication by 2^n. Putting things together, we obtain that the dual module of $(\mathbb{S}_2, \alpha)$ is given by $R_1/(u_1 - 2)$.

Since $(u_1 - 2)$ is a prime ideal in R_1, we see that the only prime associated to $R_1/(u_1 - 2)$ is $(u_1 - 2)$ itself. Noticing that the coefficient of the lowest-degree term of any element of the ideal $(u_1 - 2)$ is a multiple of two, we see that it does not contain any element of the form $u_1^n - 1$ with $n \in \mathbb{Z}$. Thus, Theorem 3.16 yields that $(\mathbb{S}_2, \alpha)$ is mixing. Further observing $\mathrm{V}_{\mathbb{C}}((u_1 - 2)) = \{2\}$ and applying Theorem 3.17, we obtain that $(\mathbb{S}_2, \alpha)$ is expansive.

We conclude this example by reflecting on the utility of the natural extension. By lifting the non-invertible base system $(\mathbb{T}, \beta)$ to the 2-adic solenoid $\mathbb{S}_2$, we transitioned to an algebraic $\mathbb{Z}$-action, thereby unlocking the dictionary developed in this chapter. While it requires some additional work, one can show that the mixing and expansive properties we just deduced for $(\mathbb{S}_2, \alpha)$ indeed imply that the original Times Two Map $(\mathbb{T}, \beta)$ is both mixing and expansive. We caution, however, that while there is a general correspondence between the mixing of the natural extension and its base action [Lac95], expansiveness of the natural extension does not *in general* imply that the base system is expansive.[7]

The preceding discussion of our four running examples illustrated the utility of the algebraic dictionary developed in this chapter for analyzing specific dynamical systems. However, the power of the dictionary does not end there. It can further be used to create a wealth of examples and counterexamples of algebraic $\mathbb{Z}^d$-actions by starting on the commutative algebra side. In this way, one can easily construct systems satisfying a specific selection of dynamical properties. The Square Shift that has been accompanying us throughout this thesis was, in fact, constructed in this way as an example of an algebraic $\mathbb{Z}^2$-action that is expansive but not mixing. As a demonstration, we will construct a few more systems in this way.

We begin by constructing examples of algebraic $\mathbb{Z}^d$-actions that are mixing but not expansive, showing that mixing and expansiveness are independent properties for general algebraic $\mathbb{Z}^d$-actions. Recall that by combining Corollary 2.7, Theorem 2.8, and Proposition 3.10, we obtain that the dual module of any expansive action must necessarily be Noetherian. This leads us to our first approach of considering non-Noetherian modules.

Example 3.25 (Rational Solenoid)**.** We consider the additive group of rational numbers $M = \mathbb{Q}$. We equip it with the structure of an R_1-module by defining $u_1^n \cdot r = 2^n r$ for any $n \in \mathbb{Z}$. The infinitude of primes implies that $\mathbb{Q}$ is not finitely generated as an R_1-module and thus not Noetherian. Indeed, let $p_1, p_2, \ldots$ be an enumeration of all odd prime numbers and consider the cyclic submodules

$$M_k = R_1 \cdot \frac{1}{p_1 \cdots p_k}$$

for all $k \geq 1$. It is readily checked that these form a strictly increasing chain of submodules. Hence, the algebraic $\mathbb{Z}$-action dual to M obtained via Proposition 3.2 is non-expansive.

Let $\mathfrak{p} \subseteq R_1$ be any prime ideal associated to M. Then $\mathfrak{p} = \mathrm{Ann}_{R_1}(r)$ for some $r \in M \setminus \{0\}$. For any $f \in R_1$, the module action gives $f \cdot r = f(2)r$. Since $r \neq 0$ and $\mathbb{Q}$ is an integral domain, $f \cdot r = 0$ if and only if $f(2) = 0$. This implies that $\mathrm{Ann}_{R_1}(r)$ is exactly the ideal $(u_1 - 2)$. Thus, the only associated prime is $\mathfrak{p} = (u_1 - 2)$.

Notice that any element $f \in (u_1 - 2)$ must satisfy $f(2) = 0$. Since $2^n - 1 = 0$ implies $n = 0$, the ideal $(u_1 - 2)$ does not contain any element of the form $u_1^n - 1$ for $n \in \mathbb{Z} \setminus \{0\}$.

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[7] For instance, one may consider the action of $\mathbb{Z}_{\geq 0}$ on $\mathbb{T}^2$ given by $n \mapsto T_A^n$, where $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is the familiar Cat Map restricted to forward time. The natural extension of this action is conjugate to the unrestricted Cat Map, which we have seen is expansive. However, due to the stable manifold induced by the eigenvalue $\frac{3-\sqrt{5}}{2}$, the restricted cat map is non-expansive.

Applying Theorem 3.16, we conclude that the algebraic $\mathbb{Z}$-action dual to M is mixing, as desired.

Next, we demonstrate that it is not necessary to require that the dual module is non-Noetherian in order for an algebraic action to be mixing and not expansive.

Example 3.26 (Mixing and non-Expansive Actions)**.** To obtain an algebraic $\mathbb{Z}^d$-action that is mixing but not expansive, it is enough to find a polynomial $f \in R_d$ that has a root in $\mathbb{S}^d$ but is not a factor of $u^{\mathbf{n}} - 1$ for any $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \{0\}$. In this case, the action dual to $R_d/(f)$ will be mixing but not expansive by Theorems 3.16 and 3.17. We provide such f for the cases $d = 1$ and $d = 2$.

For the case of $d = 1$, the minimal polynomial of any Salem number suffices. A Salem number is a real algebraic integer strictly greater than 1 whose conjugate roots have absolute values no greater than 1, with at least one having an absolute value of exactly 1. We observe that, by definition, the minimal polynomial of a Salem number cannot have a root of unity, because this would imply that it is equal to a cyclotomic polynomial whose roots all have absolute value equal to 1. The smallest *known* Salem number is the largest real root of the polynomial

$$f(u_1) = u_1^{10} + u_1^9 - u_1^7 - u_1^6 - u_1^5 - u_1^4 - u_1^3 + u_1 + 1,$$

which is also known as Lehmer's polynomial [Bor02].

We claim that $f = 1 + u_1 + u_2$ suffices in the case of $d = 2$. Setting $c_1 = \exp(2\pi i/3)$ and $c_2 = \exp(-2\pi i/3)$ and computing $f(c_1, c_2) = 0$, we see that f has roots that lie in $\mathbb{S}^2$. Assume that $u_1^{n_1} u_2^{n_2} - 1 \in (f)$ for some $(n_1, n_2) \in \mathbb{Z}^2$. Then we must have $c_1^{n_1} c_2^{n_2} = 1$ for any root (c_1, c_2) of f. Noticing that $(-2, 1)$ is a root, we obtain that $n_1 = 0$ and, by symmetry, $n_2 = 0$. This shows that (f) does not contain any polynomial of the form $u^{\mathbf{n}} - 1$ with $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus \{0\}$.

Finally, we wish to construct an algebraic $\mathbb{Z}^d$-action that is expansive but not ergodic.

Example 3.27 (Trivial Symbolic Shift)**.** Given that we want our module to be of the form $R_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq R_d$, one way to force the dual action to be expansive is to let $\mathfrak{a}$ contain an integer $k \geq 2$. To obtain non-ergodicity, we need at least one associated prime ideal to contain $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ for some $l \geq 1$. One simple way to achieve these two conditions is to set $\mathfrak{a} = (2, u_1 - 1) \subseteq R_1$. Since $(2, u_1 - 1)$ is prime, it is the only associated prime of $R_1/(2, u_1 - 1)$. Furthermore, it contains both the integer 2 and the set $\{u^n - 1 : n \in \mathbb{Z}\}$. Hence, the algebraic $\mathbb{Z}$-action dual to $R_1/(2, u_1 - 1)$ is expansive but not ergodic.

Finally, we note that in this case, the somewhat involved algebraic construction above boils down to the trivial algebraic $\mathbb{Z}$-action induced by the identity map on the discrete space $\mathbb{Z}/2\mathbb{Z}$.

Combining our examples of this section, we have shown that ergodicity, mixing, and expansiveness are independent properties for algebraic $\mathbb{Z}^d$-actions, $d \geq 1$, with the exception that mixing implies ergodicity.

Outlook

The algebraic dictionary developed in this thesis merely scratches the surface of the deep connections between the dynamical properties of algebraic $\mathbb{Z}^d$-actions and the algebraic properties of their dual modules over R_d. In his monograph *Dynamical Systems of Algebraic Origin* [Sch95], Klaus Schmidt carries this framework considerably further. For the interested reader, we briefly highlight two of the major directions in which the dictionary is extended.

Higher-Order Mixing and Rokhlin's Problem

In Chapter 1, we introduced the notion of a mixing dynamical system: any *two* events are asymptotically independent as they "move apart" in Γ. This property is also known as *2-mixing* and may be generalized to the notion of *r-mixing* for any integer $r \geq 2$ by requiring that any r events are asymptotically independent as they move apart in Γ. To be concrete, a quasi-algebraic action $\alpha : \Gamma \to \mathrm{Aut}(X)$ is said to be *r-mixing* (or *mixing of order r*) if for any measurable subsets $A_1, \ldots, A_r \subseteq X$, we have

$$\lambda_X(\alpha^{-1}_{\gamma_{1,n}}(A_1) \cap \alpha^{-1}_{\gamma_{2,n}}(A_2) \cap \cdots \cap \alpha^{-1}_{\gamma_{r,n}}(A_r)) \to \lambda_X(A_1) \cdots \lambda_X(A_r)$$

as $n \to \infty$, for any r sequences $(\gamma_{1,n})_{n \in \mathbb{N}}, \ldots, (\gamma_{r,n})_{n \in \mathbb{N}}$ in Γ such that $\gamma_{i,n}\gamma^{-1}_{j,n} \to \infty$ as $n \to \infty$ for all distinct $i, j \in \{1, \ldots, r\}$.

Let T be an automorphism[8] of a probability space $(Y, \mathcal{C}, \mu)$. We observe that the measure-preserving system $(Y, \mathcal{C}, \mu, T)$ induces a $\mathbb{Z}$-action $\beta : \mathbb{Z} \to \mathrm{Aut}(Y)$ and invite the reader to verify that the notions introduced in Section 1.2 and r-mixing make sense in this setting. In 1949, Rokhlin [Roh49] asked whether mixing implies mixing of order r for every $r \geq 2$. In this setting of a general measure-preserving system $(Y, \mathcal{C}, \mu, T)$, it is still an open problem, known as Rokhlin's Problem.[9] There are, however, classes of systems for which the question has been resolved. François Ledrappier [Led78] famously provided a counterexample demonstrating that for higher-rank actions (specifically, a $\mathbb{Z}^2$-action), 2-mixing does *not* imply 3-mixing.

Example 3.28 (Ledrappier's Three-Dot System)**.** Let σ denote the subshift of $\mathbb{Z}^2$ on the closed shift-invariant subgroup

$$X = \left\{ (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2 \right\}.$$

We observe the similarity of this system to our familiar Square Shift and recall Examples 3.8 and 3.23. From these examples, we see that the dual module of the action $\sigma : \mathbb{Z}^2 \to \mathrm{Aut}(X)$ is given by $R_2/\mathfrak{p}$, where $\mathfrak{p}$ is the prime ideal $(2, 1 + u_1 + u_2) \subseteq R_2$.

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[8] That is, T is invertible, bi-measurable and measure-preserving.

[9] We note here that Rokhlin's Problem is equivalent in the cases of T being an automorphism of $(Y, \mathcal{C}, \mu)$ and T merely being an endomorphism (i.e., a measure-preserving transformation). This is a consequence of the fact that the probabilistic notions of mixing and higher-order mixing translate between an endomorphism base system and its natural extension. The natural extension was briefly discussed in Example 3.24. For further details, we refer the reader to [Roh61; Lac95] and [CFS82, Chapter 10, §4].

We first use Theorem 3.16 to prove that σ is mixing. Assume that $u_1^{n_1}u_2^{n_2}-1 \in \mathfrak{p}$ for some $(n_1, n_2) \in \mathbb{Z}^2$. This implies that in the quotient ring $\mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]/(1+u_1+u_2)$ the equation $u_1^{n_1}u_2^{n_2} = 1$ must hold. Next, we notice that we may embed $\mathbb{F}_2[u_1^{\pm 1}, u_2^{\pm 1}]/(1+u_1+u_2)$ into the field $\mathbb{F}_2(t)$ via $u_1 \mapsto t$ and $u_2 \mapsto 1+t$. This yields the equation $t^{n_1}(1+t)^{n_2} = 1$ in $\mathbb{F}_2(t)$. By writing $n_1 = a_1 - b_1$ and $n_2 = a_2 - b_2$ for non-negative integers a_1, a_2, b_1, b_2, we obtain the equation $t^{a_1}(1+t)^{a_2} = t^{b_1}(1+t)^{b_2}$ in the unique factorization domain $\mathbb{F}_2[t]$, where t and $1+t$ are prime elements. Thus, we obtain $n_1 = n_2 = 0$ and thereby conclude that σ is mixing.

Finally, to conclude Ledrappier's counterexample, we show that σ is not 3-mixing. We begin by analyzing how the local rule in the definition of X above propagates. We claim that for any $(x_\mathbf{n}) \in X$, the equation $x_\mathbf{n} + x_{\mathbf{n}+(2^k,0)} + x_{\mathbf{n}+(0,2^k)} = 0$ holds for all $\mathbf{n} \in \mathbb{Z}^2$ and for all integers $k \geq 0$. We prove this claim by induction on k. The case $k = 0$ is clear. Let $k > 0$ and assume the claim holds for $k-1$. Let $(x_\mathbf{n}) \in X$, $\mathbf{n} \in \mathbb{Z}^2$ and observe

$$\begin{aligned} x_\mathbf{n} + x_{\mathbf{n}+(2^k,0)} + x_{\mathbf{n}+(0,2^k)} &= \Big(x_\mathbf{n} + x_{\mathbf{n}+(2^{k-1},0)} + x_{\mathbf{n}+(0,2^{k-1})}\Big) \\ &\quad + \Big(x_{\mathbf{n}+(2^{k-1},0)} + x_{\mathbf{n}+(2^k,0)} + x_{\mathbf{n}+(2^{k-1},2^{k-1})}\Big) \\ &\quad + \Big(x_{\mathbf{n}+(0,2^{k-1})} + x_{\mathbf{n}+(2^{k-1},2^{k-1})} + x_{\mathbf{n}+(0,2^k)}\Big) \\ &= 0 \end{aligned}$$

where for the second equality, we used our induction hypothesis. This proves our claim. We will now explicitly construct three measurable subsets of X that do not satisfy the 3-mixing property. Let A, B, and C all be equal to the cylinder set $\{(x_\mathbf{n})_{\mathbf{n}\in\mathbb{Z}^2} \in X : x_\mathbf{0} = 0\}$ and consider the sequences $\mathbf{n}_{1,k} = (2^k, 0)$ and $\mathbf{n}_{2,k} = (0, 2^k)$ for $k \geq 0$ in $\mathbb{Z}^2$. For any $k \geq 0$, we observe using our claim that

$$\begin{aligned} A \cap \sigma_{\mathbf{n}_{1,k}}^{-1}(B) \cap \sigma_{\mathbf{n}_{2,k}}^{-1}(C) &= \Big\{(x_\mathbf{n}) \in X : x_{(0,0)} = 0 \text{ and } x_{(2^k,0)} = 0 \text{ and } x_{(0,2^k)} = 0\Big\} \\ &= \Big\{(x_\mathbf{n}) \in X : x_{(0,0)} = 0 \text{ and } x_{(2^k,0)} = 0\Big\}, \end{aligned}$$

and hence $\lambda_X(A \cap \sigma_{\mathbf{n}_{1,k}}^{-1}(B) \cap \sigma_{\mathbf{n}_{2,k}}^{-1}(C)) = \frac{1}{2}\cdot\frac{1}{2}$ (since any two distinct coordinates are independent), while $\lambda_X(A)\lambda_X(B)\lambda_X(C) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$. This shows that σ is not 3-mixing.

For algebraic $\mathbb{Z}$-actions, Sergey Yuzvinskii [Juz65] proved that ergodicity implies mixing of all orders[10], thereby resolving Rokhlin's Problem for yet another class of systems.

Combining the previous findings of Ledrappier and Yuzvinskii, we see that Rokhlin's Problem for algebraic $\mathbb{Z}^d$-actions has a positive answer for $d = 1$ and a negative one for $d \geq 2$. However, it is not apparent *why* it breaks down for $d \geq 2$. In Chapter 8 of his monograph [Sch95], Schmidt expands the algebraic dictionary developed in this thesis to characterize higher-order mixing, thereby providing insight into the nature of this failure for $d \geq 2$. Specifically, he proves the following.[11]

Theorem 3.29 ([Sch95, Thms. 27.2 & 27.3])**.** *Let $d \geq 1$, and let M be a countable R_d-module. Then the following are equivalent for every integer $r \geq 2$.*

(a) α^M is r-mixing.

---

[10] Yuzvinskii actually proved that a continuous surjective endomorphism of a compact group is ergodic if and only if it has completely positive entropy, which implies our statement via [Sch95, Thm. 20.14].

[11] The results of Theorem 3.29 were first established as joint work by Kitchens and Schmidt [KS93], and by Schmidt and Ward [SW93].

(b) $\alpha^{R_d/\mathfrak{p}}$ *is r-mixing for every prime ideal* $\mathfrak{p}$ *associated to* M*.*

Moreover, the following hold for any prime ideal $\mathfrak{p} \subseteq R_d$*.*

(1) If $\operatorname{char}(R_d/\mathfrak{p}) > 0$*, then* $\alpha^{R_d/\mathfrak{p}}$ *is r-mixing for every* $r \geq 2$ *if and only if* $\mathfrak{p} = (p)$ *for a prime number* $p \in \mathbb{Z}$*.*

(2) If $\operatorname{char}(R_d/\mathfrak{p}) = 0$*, then* $\alpha^{R_d/\mathfrak{p}}$ *is r-mixing for every* $r \geq 2$ *if and only if* $\alpha^{R_d/\mathfrak{p}}$ *is 2-mixing.*

As a first consequence of Theorem 3.29, we see that in characteristic zero, the notions of mixing and higher-order mixing coincide. Consequently, for any algebraic $\mathbb{Z}^d$-action on a torus $\mathbb{T}^n$, mixing implies mixing of all orders.

We further get insight into how Rokhlin's proposed implication fails for $d \geq 2$. Indeed, suppose we wanted to construct an action $\alpha^{R_d/\mathfrak{p}}$ that is mixing but not mixing of all orders, for a prime ideal $\mathfrak{p} \subseteq R_d$. Then, by Theorem 3.29, we must have $\operatorname{char}(R_d/\mathfrak{p}) > 0$, i.e., $\mathfrak{p} \cap \mathbb{Z} = (p)$ for some prime number p.

In the case $d = 1$, this forces $\mathfrak{p}$ to be of the form (p) or (p, f), where f is a polynomial in u_1 that is irreducible modulo p.[12] By Theorem 3.29, we may exclude (p) and are thus left with the option $\mathfrak{p} = (p, f)$. Notice, however, that $R_1/(p, f) \cong \mathbb{F}_p[u_1^{\pm 1}]/(f) \cong \mathbb{F}_p[u_1]/(f)$ is a finite field. This implies in particular that u_1 has finite order in $R_1/(p, f)$, i.e., there exists an integer $k \geq 1$ such that $u_1^k - 1 \in (p, f)$. By Theorem 3.16, this implies that $\alpha^{R_1/(p,f)}$ is not mixing.

We thus see that it is impossible to construct a mixing action that is not mixing of all orders in the case $d = 1$. One way to view this failure is by considering the Krull dimension of the rings involved. R_1 has Krull dimension 2. If we quotient by (p) in R_1, the dimension decreases by 1, so $\mathbb{F}_p[u_1^{\pm 1}]$ has Krull dimension 1. By Theorem 3.29, we must quotient again by an irreducible polynomial in $\mathbb{F}_p[u_1^{\pm 1}]$, which results in a ring of Krull dimension 0. So in this sense, the ring R_1 is "too small" to allow for this phenomenon. Indeed, one can prove that if $S = R_d/\mathfrak{a}$ has Krull dimension 0, for some ideal $\mathfrak{a} \subseteq R_d$ and $d \geq 1$, then S and $X^S = \widehat{S}$ must be finite.[13] This makes the dynamics of (X^S, α^S) trivial: every point is α^S-periodic, α^S is trivially expansive since X^S is discrete, and it is non-ergodic since the space partitions into multiple distinct finite orbits (unless $X^S = \{0\}$).

By contrast, for $d \geq 2$, we can find prime ideals $\mathfrak{p} \subseteq R_d$ such that $R_d/\mathfrak{p}$ has Krull dimension at least 1 and $\operatorname{char}(R_d/\mathfrak{p}) > 0$. For example, the ideal $(p, 1 + u_1 + u_2)$ is prime in R_2 and $R_2/(p, 1 + u_1 + u_2)$ has Krull dimension 1. Using Theorems 3.16 and 3.29, we see that the action dual to $R_2/(p, 1 + u_1 + u_2)$ is mixing but not mixing of all orders. We invite the reader to revisit Example 3.28 at this point and observe that it does not depend on the characteristic being equal to 2.

In summary, Theorem 3.29 shows that the failure of Rokhlin's proposed implication for $d \geq 2$ can be attributed to the fact that R_d has Krull dimension $d + 1$, which allows for the existence of prime ideals $\mathfrak{p}$ such that $R_d/\mathfrak{p}$ has positive characteristic and Krull dimension at least 1, which in turn allows for the existence of mixing actions that are not mixing of all orders. In contrast, for $d = 1$, the Krull dimension of R_1 is too small to allow for this phenomenon, and thus, mixing implies mixing of all orders.

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[12] We invite the reader to check that all prime ideals in R_1 are of the form (0), or (p) for a prime number p, or (f) for an irreducible polynomial f, or (p, f) for a prime number p and a polynomial f that is irreducible modulo p. It might help to view R_1 as a localization of $\mathbb{Z}[u_1]$.

[13] One possible argument goes as follows. Since S is Noetherian and of Krull dimension 0, it is Artinian. By the structure theorem of Artinian rings, S decomposes into a finite direct sum of local Artinian rings. Since S is finitely generated as a $\mathbb{Z}$-algebra, so is each of the local Artinian components. Let $(A, \mathfrak{m})$ be a local Artinian ring that is finitely generated as a $\mathbb{Z}$-algebra. Consider the filtration $A \supseteq \mathfrak{m} \supseteq \ldots \supseteq \mathfrak{m}^k = 0$. By Zariski's lemma, $A/\mathfrak{m}$ is a finite field and thus each quotient $\mathfrak{m}^{j-1}/\mathfrak{m}^j$ is finite, implying that A is finite. Thus, S is finite.

Entropy, the Mahler Measure, and Lehmer's Problem

For an integer $k \geq 2$, let $\sigma^{(k)}$ denote the shift-action of $\mathbb{Z}$ on $(\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}}$, as per Definition 2.1. We consider the shift-actions $\sigma^{(2)}$ and $\sigma^{(3)}$. Following Example 3.8, we see that the dual module of $\sigma^{(2)}$ is $R_1/(2)$ and the dual module of $\sigma^{(3)}$ is $R_1/(3)$. Since the dual modules have distinct characteristics, it is clear that these systems are not conjugate as algebraic $\mathbb{Z}$-actions (Proposition 3.2). However, we invite the reader to check using the dictionary (Section 3.3) that these two systems are indistinguishable in terms of the dynamical properties we have discussed in this thesis (i.e., topological transitivity, ergodicity, mixing, and expansiveness).

This indistinguishability is resolved by one of the most fundamental dynamical invariants, *entropy*.[14] There are two well-known notions of dynamical entropy: *metric entropy* (also known as *Kolmogorov–Sinai entropy*) and *topological entropy*. The former is defined in the context of measurable dynamics, while the latter is defined for topological dynamical systems. We will not distinguish between these two notions, as for algebraic $\mathbb{Z}^d$-actions, the topological entropy coincides with the metric entropy with respect to the Haar measure [Sch95, Theorem 13.3]. (Recall that algebraic $\mathbb{Z}^d$-actions naturally fit into both frameworks.)

We omit a formal definition of either notion of entropy, as they are involved and unnecessary for the present discussion. We note, however, that intuitively entropy quantifies the rate at which a system generates "information" or, equivalently, the rate at which "uncertainty" grows as the system evolves over time.

Computing the entropy of a given system directly from the definition is often a challenging task. However, in a series of papers, Sinai, Rokhlin, and Yuzvinskii famously established a simple formula for computing the entropy of a toral automorphism: Let $T_B : \mathbb{T}^n \to \mathbb{T}^n$ be a toral automorphism induced by a matrix $B \in \mathrm{GL}_n(\mathbb{Z})$. Then the entropy of T_B is given by

$$h(T_B) = \sum_{|\lambda|>1} \log|\lambda|, \tag{3.4}$$

where λ ranges over the eigenvalues of B counted with multiplicity. We observe that this formula aligns with our intuition. The more B stretches the space (i.e., the larger the unstable eigenvalues), the faster T_B generates uncertainty, and thus the higher the entropy.

In a seminal paper, Lind, Schmidt, and Ward [LSW90] generalized the formula of Sinai, Rokhlin, and Yuzvinskii to the class of algebraic $\mathbb{Z}^d$-actions by using the deep connections between the dynamics and commutative algebra we have explored in this thesis.

To make the connection to the commutative algebra side in the case of the toral automorphism more apparent, recall from Example 3.7 that the dual module of T_B is given by $R_1/(f_B)$, where f_B is the characteristic polynomial of B. Then, using Jensen's formula (cf. [Sch95, Proposition 16.1]), the sum on the right-hand side of Eq. (3.4) is seen to be equal to the logarithm of the so-called *Mahler measure* of f_B. For any integer $d \geq 1$ and any polynomial $f \in R_d$, the Mahler measure of f is defined as

$$\mathbb{M}(f) = \exp\left(\int_{\mathbb{S}^d} \log|f(\mathbf{s})|\, d\lambda_{\mathbb{S}^d}(\mathbf{s})\right)$$

if $f \neq 0$ and $\mathbb{M}(f) = 0$ otherwise. Here, $\mathbb{S}^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$ and $\lambda_{\mathbb{S}^d}$ denotes the normalized Haar measure of the compact abelian group $\mathbb{S}^d$. Using this connection, Lind, Schmidt, and Ward established the following theorem.

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[14] Historically, the narrative was reversed: for a long time, it was not known whether $\sigma^{(2)}$ and $\sigma^{(3)}$ are *measurably conjugate* or not. This remained an open question until the introduction of entropy by Kolmogorov. Entropy is invariant under measurable conjugacy and the entropies of $\sigma^{(2)}$ and $\sigma^{(3)}$ are distinct.

Theorem 3.30 ([Sch95, Theorem 18.1])**.** *Let* $d \geq 1$. *Then*

$$h(\alpha^{R_d/(f)}) = |\log \mathbb{M}(f)|$$

for every $f \in R_d$.

We observe that using this theorem, the entropy of the shift-action $\sigma^{(k)}$ is given by $|\log \mathbb{M}(k)| = \log(k)$, thus resolving the indistinguishability of $\sigma^{(2)}$ and $\sigma^{(3)}$ mentioned at the beginning of this subsection.

By analyzing the entropy of actions dual to $R_d/\mathfrak{p}$ for primes $\mathfrak{p}$, Lind, Schmidt, and Ward further established:

Theorem 3.31 ([Sch95, Proposition 18.6])**.** *Let* $d \geq 1$*, and let* M *be a Noetherian* R_d*-module. Then*

$$h(\alpha^M) = \sum_{\mathfrak{q}} h(\alpha^{R_d/\mathfrak{q}}),$$

where $\mathfrak{q}$ *ranges over the prime ideals occurring in a prime filtration of* M*, and*

$$h(\alpha^{R_d/\mathfrak{p}}) = \begin{cases} |\log \mathbb{M}(f)| & \text{if } \mathfrak{p} = (f) \text{ is principal,} \\ 0 & \text{if } \mathfrak{p} \text{ is not principal,} \end{cases}$$

for every prime ideal $\mathfrak{p} \subseteq R_d$.

Moreover, if N *is an arbitrary countable* R_d*-module, then there exists an increasing sequence* $(N_k)_{k\geq 1}$ *of Noetherian submodules of* N *such that* $N = \bigcup_{k\geq 1} N_k$ *and we have* $h(\alpha^N) = \lim_{k\to\infty} h(\alpha^{N_k})$.

This connection between dynamical entropy and the Mahler measure has created a link between dynamics and a long-standing conjecture due to Lehmer [Leh33]:

Conjecture 3.32 (Lehmer's Conjecture)**.** *There exists a constant* $c > 1$ *such that every irreducible non-cyclotomic polynomial* $f \in \mathbb{Z}[u] \setminus \{\pm u\}$ *satisfies* $\mathbb{M}(f) \geq c$.

The smallest known Mahler measure greater than 1 belongs to *Lehmer's polynomial*:

$$L(u) = u^{10} + u^9 - u^7 - u^6 - u^5 - u^4 - u^3 + u + 1,$$

with $\mathbb{M}(L) \approx 1.17628$. Despite nearly a century of effort, no polynomial with Mahler measure strictly between 1 and $\mathbb{M}(L)$ has been found, nor has anyone proved that such a polynomial does not exist.

Through Theorem 3.30 (and Theorem 3.12), Lehmer's conjecture may be translated into the realm of dynamical systems.

Conjecture 3.33 (Dynamical Lehmer Conjecture)**.** *There exists a constant* $\varepsilon > 0$ *such that every ergodic algebraic* $\mathbb{Z}$*-action* α *with* $h(\alpha) > 0$ *satisfies* $h(\alpha) \geq \varepsilon$.

This deep connection to number theory illustrates that the algebraic dictionary developed by Klaus Schmidt and others is not merely a useful tool to study and characterize dynamical systems. Rather, it is a profound bridge between distinct areas of mathematics.

Appendix A

Topological Groups

The results presented in this appendix and their proofs may be found in [DE14], [Fol16], and [HR79; HR70]. The structure of this appendix is inspired by the one given in [EW11].

A.1 Basic Notions

A *topological group* is a group G equipped with a topology with respect to which the group operation $G \times G \to G, (g,h) \mapsto gh$ and the inversion map $G \to G, g \mapsto g^{-1}$ are continuous, where $G \times G$ is equipped with the product topology. From the interplay between the algebraic and topological structures alone, one can already derive many interesting properties of topological groups:

Proposition A.1. *Let G be a topological group. Then the following hold:*

(a) The topology of G is invariant under translation and inversion. That is, if $U \subseteq G$ is open, then so are gU, Ug, and U^{-1} for any $g \in G$.

(b) For any neighborhood of the identity U, there exists a symmetric *neighborhood of the identity V (that is, $V = V^{-1}$) such that $VV \subseteq U$.*

(c) For compact subsets $A, B \subseteq G$, the set AB is also compact.

(d) For subsets $A, B \subseteq G$ such that A is closed and B is compact, AB is closed.

(e) For subsets $A, B \subseteq G$ such that at least one of them is open, AB is open.

(f) Any open subgroup of G is also closed.

(g) For a (normal) subgroup $H \subseteq G$, the closure $\overline{H}$ is a (normal) subgroup of G.

(h) If G is T_1, then it is Hausdorff.

(i) For a subgroup H, the canonical projection $\pi : G \to G/H$ is an open map, and G/H is T_1 if and only if H is closed.

(j) For a normal subgroup H, the quotient G/H is again a topological group.

(k) The closure of the identity $\overline{\{1_G\}}$ is the smallest closed normal subgroup, and the quotient $G/\overline{\{1_G\}}$ is a Hausdorff topological group.

A.2 Locally Compact Groups & Haar Measures

One class of topological groups that is of particular interest to us is the class of locally compact groups. A *(locally) compact group* is a topological group G whose underlying topological space is Hausdorff and (locally) compact. Examples of locally compact groups include:

- The real line $\mathbb{R}$ as an additive group with the usual topology.
- The 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the quotient topology.
- The integers $\mathbb{Z}$ equipped with the discrete topology (indeed, any group equipped with the discrete topology forms a locally compact group).
- The general linear group $\mathrm{GL}_n(\mathbb{R})$ of $n \times n$ invertible matrices over $\mathbb{R}$ equipped with the subspace topology inherited from $\mathbb{R}^{n^2}$.

There is a useful characterization, due to Birkhoff and Kakutani, of when a locally compact group admits a compatible metric.

Theorem A.2. *Let G be a locally compact group. Then the following are equivalent:*

(a) G is first-countable.
(b) G is metrizable (as a topological space).
(c) G admits a left-invariant compatible metric, that is, a metric $d : G \times G \to \mathbb{R}_{\geq 0}$ such that $d(gx, gy) = d(x, y)$ for all $g, x, y \in G$.
(d) G admits a right-invariant compatible metric (defined analogously to a left-invariant metric).

One of the most important features of locally compact groups is the existence of a translation-invariant measure that is unique up to scaling. More concretely, for a locally compact group G, a *left Haar measure* on G is a non-zero Radon measure μ on G that satisfies $\mu(gA) = \mu(A)$ for every Borel subset $A \subseteq G$ and every $g \in G$.

Theorem A.3. *Any locally compact group G admits a left Haar measure, which is unique up to multiplication by a positive constant. Furthermore, the following properties hold:*

(a) Any compact subset of G has finite Haar measure.
(b) Any non-empty open subset of G has positive Haar measure.
(c) G is compact if and only if G has finite Haar measure.

Let G be a locally compact group and let μ be a left Haar measure on G. For any $g \in G$, one may define a measure μ_g by setting $\mu_g(A) = \mu(Ag)$ for all Borel sets $A \subseteq G$. Then μ_g is again a left Haar measure on G and thus there exists a positive constant $\Delta(g)$ such that $\mu_g = \Delta(g) \cdot \mu$. The map $\Delta : G \to \mathbb{R}_{>0}$ is called the *modular function* of G. If Δ is the trivial homomorphism, i.e., $\Delta \equiv 1$, then we say that G is *unimodular*.

Proposition A.4. *The following hold:*

(a) The modular function $\Delta : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism.
(b) If G is either abelian or compact, then G is unimodular.
(c) For any $h \in G$ and $f \in L^1(G)$, we have

$$\int_G f(gh)\, d\mu(g) = \Delta(h^{-1}) \int_G f(g)\, d\mu(g).$$

(d) For any $f \in L^1(G)$, we have

$$\int_G f(g^{-1})\Delta(g^{-1})\, d\mu(g) = \int_G f(g)\, d\mu(g).$$

A.3 Pontryagin Duality

In this final section, we restrict our attention even further to the class of locally compact *abelian* groups, also called LCA groups.

Let A be an LCA group. We define the *dual group* of A, denoted by $\widehat{A}$, as the set of all continuous group homomorphisms $\chi : A \to \mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. We also call these homomorphisms *characters* of A. As suggested by the name, the dual group forms a group under pointwise multiplication. The value of $\chi \in \widehat{A}$ at $a \in A$ will often be denoted by $\langle a, \chi \rangle$ to emphasize the symmetric nature of the spaces A and $\widehat{A}$ (established by Pontryagin duality). Furthermore, $\widehat{A}$ may be viewed as a subset of $C(A)$, the space of all continuous complex-valued functions on A. When $C(A)$ is equipped with the compact-open topology, $\widehat{A}$ inherits the subspace topology, making it a topological group.

Proposition A.5. *The dual group $\widehat{A}$, viewed as a topological group as described above, is again an LCA group.*

Since the dual group $\Sigma := \widehat{A}$ is again an LCA group, we may consider its dual group $\widehat{\Sigma}$. As suggested by the notation $\langle x, \chi \rangle$ for $x \in A$ and $\chi \in \widehat{A}$, there is a natural map

$$\begin{aligned} \Phi : A &\to \widehat{\Sigma} \\ x &\mapsto (\chi \mapsto \langle x, \chi \rangle). \end{aligned}$$

Theorem A.6 (Pontryagin Duality)**.** *The map Φ is an isomorphism of LCA groups, i.e., Φ is a continuous group isomorphism with continuous inverse.*

From this point on, we will implicitly identify the double dual $\widehat{\Sigma}$ with A.

Using Pontryagin duality, we can also relate the subgroup structure of A and its dual. We note here that a closed subgroup $B \subseteq A$ and the quotient A/B are again LCA groups.

Proposition A.7. *There is an inclusion-reversing bijection between the closed subgroups of A and those of $\widehat{A}$, given by mapping a closed subgroup to its* annihilator*. The annihilator of a closed subgroup B of A is defined as*

$$B^\perp = \{\chi \in \widehat{A} : \langle b, \chi \rangle = 1 \text{ for all } b \in B\}$$

and is a closed subgroup of $\widehat{A}$. The map $B \mapsto B^\perp$ is a bijection and its inverse is given by the analogous annihilator map from $\widehat{A}$ back to A. In particular, we have $(B^\perp)^\perp = B$ for any closed subgroup B of A.

Moreover, we have the following isomorphisms:

(a) $\widehat{A/B} \cong B^\perp$,
(b) $\widehat{A}/B^\perp \cong \widehat{B}$,
(c) $\overline{B_1^\perp + B_2^\perp} \cong \widehat{A/(B_1 \cap B_2)}$

for closed subgroups $B, B_1, B_2 \subseteq A$.

There are also useful isomorphisms for direct sums and products.

Proposition A.8. *Let $A_1, \ldots, A_n$ be LCA groups and set $A = \prod_{j=1}^n A_j$. Then we have the identification*

$$\widehat{A} \cong \prod_{j=1}^{n} \widehat{A}_j$$

such that $\langle (x_j), (\chi_j) \rangle = \prod_{j=1}^{n} \langle x_j, \chi_j \rangle$ *for* $(x_j) \in A$ *and* $(\chi_j) \in \widehat{A}$.

Let $(G_j)_{j \in J}$ *be a family of compact abelian groups and set* $G = \prod_{j \in J} G_j$. *Then we have the identification*

$$\widehat{G} \cong \bigoplus_{j \in J} \widehat{G}_j$$

such that $\langle (g_j), (\xi_j) \rangle = \prod_{j \in J} \langle g_j, \xi_j \rangle$ *for* $(g_j) \in G$ *and* $(\xi_j) \in \widehat{G}$, *where we note that all but finitely many factors in the product are equal to* 1.

Since the topology on $\widehat{A}$ inherently depends on the topology on A, one might expect that there are some relations between the topological properties of A and $\widehat{A}$.

Proposition A.9. *The following dualities hold.*

- *(a)* A *is compact if and only if* $\widehat{A}$ *is discrete.*
- *(b)* A *is second-countable if and only if* $\widehat{A}$ *is second-countable.*
- *(c) If* A *is compact, then* A *is metrizable if and only if* $\widehat{A}$ *is countable.*

Using standard tools from analysis, one can compute many dual groups explicitly. We list a few of them here.

Proposition A.10. *We have the following dualities, where elements of* $\mathbb{T}$ *and* $\mathbb{Z}/k\mathbb{Z}$ *are identified with their standard representatives in* $[0,1) \subseteq \mathbb{R} \subseteq \mathbb{C}$ *and* $\{0, 1, \ldots, k-1\} \subseteq \mathbb{Z} \subseteq \mathbb{C}$, *respectively, to evaluate the exponential.*

- *(a)* $\widehat{\mathbb{R}} \cong \mathbb{R}$ *with* $\langle x, \xi \rangle = \exp(2\pi i x \xi)$.
- *(b)* $\widehat{\mathbb{T}} \cong \mathbb{Z}$ *with* $\langle \alpha, n \rangle = \exp(2\pi i n \alpha)$.
- *(c)* $\widehat{\mathbb{Z}} \cong \mathbb{T}$ *with* $\langle n, \alpha \rangle = \exp(2\pi i n \alpha)$.
- *(d)* $\widehat{\mathbb{Z}/k\mathbb{Z}} \cong \mathbb{Z}/k\mathbb{Z}$ *for any integer* $k \geq 2$ *with* $\langle m, n \rangle = \exp(2\pi i n m / k)$.

Using Pontryagin duality, we may also dualize morphisms of LCA groups.

Proposition A.11. *Let* A *and* B *be LCA groups and* $\varphi : A \to B$ *a continuous group homomorphism. Then the map* $\hat{\varphi} : \widehat{B} \to \widehat{A}$ *defined by* $\eta \mapsto \eta \circ \varphi$ *is a continuous group homomorphism, called the* dual homomorphism *of* φ. *Moreover, the following hold:*

- *(a) The dual homomorphism* $\hat{\varphi}$ *is injective if and only if* φ *has dense image.*
- *(b) The dual homomorphism* $\hat{\varphi}$ *is an open surjection if and only if* φ *is a closed injection.*

In particular, if A *and* B *are either both compact or both discrete, then* φ *is injective if and only if* $\hat{\varphi}$ *is surjective, and* φ *is surjective if and only if* $\hat{\varphi}$ *is injective.*

Appendix B

Unitary Representations

The results presented in this appendix and their proofs may be found in standard textbooks on harmonic analysis, such as [DE14], [Fol16], and [HR79; HR70].

B.1 Basic Notions

Let G be a locally compact group.

Definition B.1. A *unitary representation* of G is a continuous group homomorphism $\tau : G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is a non-zero complex Hilbert space and $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ equipped with the strong operator topology. The *dimension* of τ is defined as the dimension of the *representation space* $\mathcal{H}$. We often denote the unitary representation $\tau : G \to \mathcal{U}(\mathcal{H})$ by the pair $(\tau, \mathcal{H})$.

The strong operator topology on the group of unitary operators $\mathcal{U}(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$ may be defined as follows: a net $(U_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathcal{U}(\mathcal{H})$ converges to $U \in \mathcal{U}(\mathcal{H})$ if and only if $\|U_\alpha(f) - U(f)\| \to 0$ for all $f \in \mathcal{H}$.

Continuity of a group homomorphism $\tau : G \to \mathcal{U}(\mathcal{H})$ with respect to the strong operator topology on $\mathcal{U}(\mathcal{H})$ is equivalent to the map $g \mapsto \tau(g)f$ being continuous from G to $\mathcal{H}$ for every $f \in \mathcal{H}$.

Definition B.2. Let $\tau : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G.

- A closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is called *invariant* if $\tau(g)\mathcal{K} \subseteq \mathcal{K}$ for all $g \in G$.
- A unitary representation $(\rho, \mathcal{F})$ is a *subrepresentation* of $(\tau, \mathcal{H})$ if $\mathcal{F}$ is an invariant closed subspace of $\mathcal{H}$ and $\rho(g)$ is the restriction of $\tau(g)$ to $\mathcal{F}$ for all $g \in G$.
- The representation τ is called *irreducible* if it does not possess any non-trivial proper subrepresentations, i.e., for every closed subspace $\mathcal{K} \subseteq \mathcal{H}$ that is invariant, one has $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$.

Given a family of unitary representations $(\tau_\alpha, \mathcal{H}_\alpha)_{\alpha\in\mathcal{A}}$, we may consider the complex Hilbert space given by the Hilbert direct sum

$$\mathcal{H} = \bigoplus_{\alpha\in\mathcal{A}} \mathcal{H}_\alpha = \left\{ (f_\alpha)_{\alpha\in\mathcal{A}} : f_\alpha \in \mathcal{H}_\alpha \text{ for all } \alpha \in \mathcal{A}, \sum_{\alpha\in\mathcal{A}} \|f_\alpha\|^2 < \infty \right\}.$$

Then we can define a unitary representation $\tau : G \to \mathcal{U}(\mathcal{H})$ by setting $\tau(g)(f_\alpha)_{\alpha\in\mathcal{A}} = (\tau_\alpha(g)f_\alpha)_{\alpha\in\mathcal{A}}$ for all $g \in G$ and $(f_\alpha)_{\alpha\in\mathcal{A}} \in \mathcal{H}$. We sometimes call τ the *direct sum* of the family of representations $(\tau_\alpha, \mathcal{H}_\alpha)_{\alpha\in\mathcal{A}}$ and write $\tau = \bigoplus_{\alpha\in\mathcal{A}} \tau_\alpha$. In this case, each $(\tau_\alpha, \mathcal{H}_\alpha)$ is a subrepresentation of $(\tau, \mathcal{H})$.

Definition B.3. Let $(\tau_1, \mathcal{H}_1)$ and $(\tau_2, \mathcal{H}_2)$ be unitary representations of G.

- A continuous linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called an *intertwining operator* or a G-homomorphism if the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "T"] \arrow[d, "\tau_1(g)"'] & \mathcal{H}_2 \arrow[d, "\tau_2(g)"] \\ \mathcal{H}_1 \arrow[r, "T"] & \mathcal{H}_2 \end{tikzcd}$$

  for every $g \in G$. We write $\mathrm{Hom}_G(\tau_1, \tau_2)$ for the set of all G-homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$.
- We say that $(\tau_1, \mathcal{H}_1)$ and $(\tau_2, \mathcal{H}_2)$ are *(unitarily) equivalent*, denoted by $\tau_1 \simeq \tau_2$, if there exists a unitary intertwining operator $U : \mathcal{H}_1 \to \mathcal{H}_2$.
- We denote by $\widehat{G}$ the set of all equivalence classes of irreducible unitary representations of G. We call $\widehat{G}$ the *unitary dual* of G.

Lemma B.4 (Schur's Lemma)**.** *The following hold:*

(a) A unitary representation $(\tau, \mathcal{H})$ of G is irreducible if and only if $\mathrm{Hom}_G(\tau, \tau)$ *contains only scalar multiples of the identity, i.e.,* $\mathrm{Hom}_G(\tau, \tau) = \mathbb{C} \cdot \mathrm{id}_{\mathcal{H}}$.

(b) Suppose τ_1 and τ_2 are irreducible unitary representations of G. If τ_1 and τ_2 are equivalent, then $\mathrm{Hom}_G(\tau_1, \tau_2)$ *is one-dimensional. Otherwise,* $\mathrm{Hom}_G(\tau_1, \tau_2) = \{0\}$.

Suppose that G is abelian, and let $(\tau, \mathcal{H})$ be an irreducible unitary representation of G. Then we have

$$\tau(g)\tau(h) = \tau(gh) = \tau(h)\tau(g),$$

for any $g, h \in G$. This shows that, for every $h \in G$, $\tau(h)$ is an intertwining operator for τ. By Schur's lemma (Lemma B.4), we obtain that $\tau(h) = \lambda_h \,\mathrm{id}_{\mathcal{H}}$ for some $\lambda_h \in \mathbb{C}$. This in turn implies that any closed one-dimensional subspace of $\mathcal{H}$ is invariant. By irreducibility, we conclude that $\mathcal{H}$ is one-dimensional.

It follows from the above that if G is abelian, then any irreducible representation may be viewed as a continuous group homomorphism $\tau : G \to \mathcal{U}(\mathbb{C}) \cong \mathbb{S}^1$. Hence, the unitary dual $\widehat{G}$ coincides with the dual group of G from Appendix A, justifying the notation. In this sense, irreducible unitary representations generalize the notion of characters to the non-abelian case.

B.2 The Right Regular Representation

Let X be a compact group and let λ_X be the normalized Haar measure on X. We can define a unitary representation $\rho : X \to \mathcal{U}(L^2(X, \lambda_X))$ by setting $(\rho(x)f)(y) = f(yx)$ for all $x, y \in X$ and $f \in L^2(X, \lambda_X)$. This representation is called the *right regular representation* of X.[1] To see that ρ is indeed a unitary representation, one may check that $\rho(x)$ is a unitary operator by using unimodularity of X (see Proposition A.4) and noting that $\rho(x^{-1})$ defines the inverse of $\rho(x)$. For continuity of ρ, one can first prove that $x \mapsto \rho(x)f$ is continuous for continuous functions and then extend to all of $L^2(X, \lambda_X)$ via an approximation argument.

We define

$$L^2_0(X, \lambda_X) = \left\{ f \in L^2(X, \lambda_X) : \int_X f \, d\lambda_X = 0 \right\},$$

---

[1] One may similarly define the left regular representation of X by $(\sigma(x)f)(y) = f(x^{-1}y)$. We will restrict our attention to the right regular representation.

and notice that, as the kernel of the continuous linear functional $f \mapsto \langle f, \mathbb{1}_X \rangle$, it is a closed subspace of $L^2(X, \lambda_X)$. We further observe that it is the orthogonal complement of the one-dimensional closed subspace $\mathbb{C} \cdot \mathbb{1}_X$ of constant functions. Since these subspaces are both invariant under ρ, we see that ρ decomposes into subrepresentations on $L^2_0(X, \lambda_X)$ and $\mathbb{C} \cdot \mathbb{1}_X$.

B.3 The Peter–Weyl Theorem

Before we state the Peter–Weyl theorem, we must define the functions that will serve as our fundamental building blocks. Fix a compact group X.

Definition B.5. Let $(\tau, \mathcal{H})$ be a unitary representation of X. A *matrix coefficient* of τ is a function $f_{v,w} : X \to \mathbb{C}$ of the form $f_{v,w}(x) = \langle \tau(x)v, w \rangle$ for some $v, w \in \mathcal{H}$.

Let $\widehat{X}_{\text{fin}}$ denote the subset of the unitary dual $\widehat{X}$ consisting of equivalence classes of *finite-dimensional* irreducible unitary representations. For each class in $\widehat{X}_{\text{fin}}$, we fix a representative $(\tau, \mathcal{H}_\tau)$. Since each representative $(\tau, \mathcal{H}_\tau)$ is finite-dimensional, we may choose an orthonormal basis $e_1, \dots, e_{\dim \mathcal{H}_\tau}$ of $\mathcal{H}_\tau$ and define $\tau_{ij}(x) = \langle \tau(x)e_i, e_j \rangle$. The function $\tau_{ij} : X \to \mathbb{C}$ is called the (i, j)-th matrix coefficient of the representation τ.

There are many formulations of the Peter–Weyl theorem, each useful for different purposes. We provide two such formulations that fit our use cases.

Theorem B.6 (Peter–Weyl I)**.** *The family*

$$\left(\sqrt{\dim \mathcal{H}_\tau} \tau_{ij} \right)_{\tau, i, j},$$

where τ ranges over the fixed representatives of $\widehat{X}_{\text{fin}}$ and i, j range over $\{1, \dots, \dim \mathcal{H}_\tau\}$, forms a Hilbert basis of $L^2(X, \lambda_X)$.

This formulation is particularly useful when X is abelian. In this case, all irreducible unitary representations are one-dimensional, and the family of matrix coefficients reduces to the family of characters of X. Consequently, the dual group $\widehat{X}$ forms a Hilbert basis of $L^2(X, \lambda_X)$.

Theorem B.7 (Peter–Weyl II)**.** *Let $(\rho, L^2(X, \lambda_X))$ be the right regular representation of X. Then we have the unitary equivalence*

$$\rho \simeq \bigoplus_{\tau \in \widehat{X}_{\text{fin}}} \tau^{\oplus \dim \mathcal{H}_\tau}.$$

There are a few corollaries of the Peter–Weyl theorem that will be useful to us.

Corollary B.8. *The following hold:*

- *(a) Every irreducible unitary representation of X is finite-dimensional, i.e., $\widehat{X} = \widehat{X}_{\text{fin}}$.*
- *(b) Every unitary representation of X is a direct sum of irreducible unitary representations of X.*
- *(c) The points of X are separated by $\widehat{X}$. That is, for any two distinct points $x, y \in X$, there exists an irreducible unitary representation τ of X such that $\tau(x) \neq \tau(y)$.*

Appendix C

Commutative Algebra

The purpose of this appendix is to introduce some fundamental concepts from commutative algebra upon which the present thesis relies. These include, in particular, Noetherian rings and modules, associated primes and primary decomposition, and Hilbert's Nullstellensatz.

Everything presented in this appendix is standard and may be found in [Eis95] and [AM69].

C.1 Noetherian Rings & Modules

Fix a commutative ring R with unit and an R-module M.

Definition C.1. We call M *Noetherian* if every ascending chain of submodules of M stabilizes, i.e., for every sequence of submodules $M_1 \subseteq M_2 \subseteq \dots$, there exists an integer $K \geq 1$ such that $M_k = M_K$ for all $k \geq K$. We say that the ring R is *Noetherian* if it is Noetherian when viewed as a module over itself.

The Noetherian property is best thought of as a finiteness condition. The following proposition makes this precise.

Proposition C.2. *The following conditions on M are equivalent:*

- *(a) M is Noetherian.*
- *(b) Every non-empty set of submodules of M contains a maximal element with respect to inclusion.*
- *(c) Every submodule of M is finitely generated.*

Proposition C.3. *If R is Noetherian and M is finitely generated over R, then M is Noetherian.*

The following five lemmas are fundamental finiteness results that we rely on in the main text.

Lemma C.4 (Cayley–Hamilton Theorem)**.** *Suppose M is finitely generated as an R-module, say by n elements. Let $\mathfrak{a} \subseteq R$ be an ideal and let $\varphi : M \to M$ be an R-linear endomorphism such that $\varphi(M) \subseteq \mathfrak{a}M$. Then φ satisfies a monic polynomial equation of the form*

$$\varphi^n + a_{n-1}\varphi^{n-1} + \dots + a_1\varphi + a_0 = 0$$

where each coefficient a_j lies in $\mathfrak{a}^{n-j}$ for all $j \in \{0, \dots, n-1\}$.

Lemma C.5 (Nakayama's Lemma)**.** *Suppose M is finitely generated as an R-module and $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a}M = M$. Then there exists an element $a \in \mathfrak{a}$ such that $(a+1)M = 0$.*

Lemma C.6 (Artin–Tate Lemma)**.** *Let $A \subseteq B \subseteq C$ be commutative rings. Assume that A is Noetherian, that C is a finitely generated A-algebra, and that C is a finitely generated B-module. Then B is a finitely generated A-algebra.*[1]

Lemma C.7 (Artin–Rees Lemma)**.** *Suppose R is Noetherian and M is finitely generated over R. If $\mathfrak{a} \subseteq R$ is an ideal and $N \subseteq M$ is a submodule, then there exists an integer $k \geq 1$ such that*

$$\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-k}(\mathfrak{a}^k M \cap N)$$

for all $n \geq k$.

Lemma C.8 (Zariski's Lemma)**.** *Let k be a field and let E be a finitely generated k-algebra. If E is a field, then it is a finite algebraic extension of k.*

C.2 Hilbert's Nullstellensatz

Let k be an algebraically closed field, and let $k[X_1, \ldots, X_d]$ be the polynomial ring in d variables over k. For any subset $S \subseteq k[X_1, \ldots, X_d]$, we define the zero set $\mathrm{V}(S)$ of S as

$$\mathrm{V}(S) = \Big\{(x_1, \ldots, x_d) \in k^d : f(x_1, \ldots, x_d) = 0 \text{ for all } f \in S\Big\}.$$

Conversely, given a subset $A \subseteq k^d$, we define the ideal $\mathrm{I}(A)$ of polynomials vanishing on A as

$$\mathrm{I}(A) = \{f \in k[X_1, \ldots, X_d] : f(x_1, \ldots, x_d) = 0 \text{ for all } (x_1, \ldots, x_d) \in A\}.$$

Hilbert's Nullstellensatz provides a precise relation between these two operations. Before we state it, recall that given an ideal $\mathfrak{a} \subseteq k[X_1, \ldots, X_d]$, the *radical* of $\mathfrak{a}$ is defined as

$$\mathrm{rad}(\mathfrak{a}) = \{f \in k[X_1, \ldots, X_d] : f^m \in \mathfrak{a} \text{ for some } m \geq 1\}.$$

Theorem C.9 (Hilbert's Nullstellensatz)**.** *For any ideal $\mathfrak{a} \subseteq k[X_1, \ldots, X_d]$, we have*

$$\mathrm{I}(\mathrm{V}(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a}).$$

While Theorem C.9 is stated for the polynomial ring $k[X_1, \ldots, X_d]$ and varieties in the affine space k^d, we often consider the ring of Laurent polynomials $k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ and varieties in the algebraic torus $(k^\times)^d$. This gap can be bridged by identifying the ring of Laurent polynomials with the quotient

$$k[X_1, Y_1, \ldots, X_d, Y_d]/(X_1Y_1 - 1, \ldots, X_dY_d - 1).$$

Under this identification, the algebraic torus $(k^\times)^d$ corresponds to the zero locus of the polynomials $X_iY_i - 1$ in k^{2d}. Consequently, Hilbert's Nullstellensatz applies directly to ideals in the ring of Laurent polynomials by viewing them as ideals in $k[X_1, Y_1, \ldots, X_d, Y_d]$ containing the polynomials $X_iY_i - 1$, for $i = 1, \ldots, d$. The following proposition will also be useful to us.

---

[1] For the reader unfamiliar with algebras, we note that C being finitely generated as an A-algebra is equivalent to the existence of a surjective ring homomorphism from a polynomial ring $A[X_1, \ldots, X_d]$ to C for some integer $d \geq 1$.

Proposition C.10. *Given a non-zero, non-unit polynomial* $f \in k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ *for* $d \geq 2$, *the variety* $\mathrm{V}(f) \subseteq (k^\times)^d$ *is infinite.*

This essentially follows from the intuitive observation that by introducing a single constraint, namely the polynomial equation $f = 0$, the dimension of $(k^\times)^d$ is reduced by at most one. Consequently, $\mathrm{V}(f)$ has dimension $\geq d - 1 \geq 1$ and is thus infinite.

C.3 Associated Primes & Primary Decomposition

Historically, the theory of primary decomposition was developed by Emanuel Lasker and Emmy Noether to generalize the notion of unique prime factorization in $\mathbb{Z}$ to more general rings. While it was first formulated only for ideals in rings, the theory was later generalized to modules using the notion of associated prime ideals [Noe21; Las05; Eis95].

Again, we fix a commutative ring R with unit and an R-module M.

Definition C.11. A prime ideal $\mathfrak{p}$ is *associated to* M if $\mathfrak{p}$ is the annihilator of a non-zero element of M, i.e., if there exists a non-zero element $m \in M$ such that $\mathfrak{p} = \{r \in R : rm = 0\}$. The set of all primes associated to M is denoted by $\mathrm{Ass}_R(M)$, or simply $\mathrm{Ass}(M)$ when the underlying ring is clear.

Proposition C.12. *Suppose we have a short exact sequence*

$$0 \to A \to B \to C \to 0$$

of R*-modules. Then* $\mathrm{Ass}(A) \subseteq \mathrm{Ass}(B)$ *and* $\mathrm{Ass}(B) \subseteq \mathrm{Ass}(A) \cup \mathrm{Ass}(C)$. *Furthermore, we have* $\mathrm{Ass}(A \oplus C) = \mathrm{Ass}(A) \cup \mathrm{Ass}(C)$.

The next proposition establishes how the associated primes are, in a sense, the building blocks of the module.

Proposition C.13. *Assume that both* R *and* M *are Noetherian. Then* M *admits a filtration (often called a* prime filtration *of* M*)*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $M_j / M_{j-1} \cong R/\mathfrak{p}_j$ *for some prime ideal* $\mathfrak{p}_j$ *for each* $j \in \{1, \dots, n\}$. *Furthermore, we have that* $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ *and for each* $j \in \{1, \dots, n\}$, *there exists a* $\mathfrak{p} \in \mathrm{Ass}(M)$ *such that* $\mathfrak{p} \subseteq \mathfrak{p}_j$.

Proposition C.14. *Suppose that both* R *and* M *are Noetherian and that* M *is non-zero. Then the following hold:*

(a) The set of associated primes $\mathrm{Ass}(M)$ *is finite and non-empty.*
(b) Each prime in $\mathrm{Ass}(M)$ *contains* $\mathrm{Ann}(M)$.
(c) The set $\mathrm{Ass}(M)$ *contains every prime ideal of* R *minimal among those containing* $\mathrm{Ann}(M)$.
(d) The union of the associated primes of M *is exactly the set of zero divisors on* M.

Having established the notion of prime ideals associated to a module, we now turn to primary decomposition. For the remainder of this appendix, we will always assume that our fixed ring R and module M are both Noetherian.

Definition C.15. A proper submodule N of M is called *primary* if $\mathrm{Ass}(M/N)$ consists of exactly one prime ideal. If $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, then we say that N is $\mathfrak{p}$*-primary*.

A fundamental property of primary submodules is that they are preserved under finite intersections.

Proposition C.16. *Let $\mathfrak{p}$ be a prime ideal of R, and suppose $N_1, \ldots, N_k$ are $\mathfrak{p}$-primary submodules of M. Then their intersection $\bigcap_{j=1}^{k} N_j$ is again a $\mathfrak{p}$-primary submodule of M.*

The final theorem of this appendix is concerned with the existence of a unique (in the sense described below) primary decomposition, which generalizes the Fundamental Theorem of Arithmetic (i.e., unique prime factorization in $\mathbb{Z}$).

Definition C.17. Let $N \subsetneq M$ be a proper submodule. A *primary decomposition* of N is an expression of N as a finite intersection of primary submodules

$$N = Q_1 \cap \cdots \cap Q_m.$$

Such a decomposition is called *minimal* if no Q_i can be omitted from the intersection, that is, $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$ for all $i \in \{1, \ldots, m\}$, and the associated prime ideals $\mathfrak{p}_i$ of the quotients M/Q_i are all distinct.

Notice that if a submodule $N \subsetneq M$ admits a primary decomposition, we may always reduce to a minimal primary decomposition by dropping unnecessary terms and using Proposition C.16.

Theorem C.18 (Lasker–Noether)**.** *Every proper submodule $N \subsetneq M$ admits a minimal primary decomposition $N = \bigcap_{j=1}^{m} Q_j$ such that $\operatorname{Ass}(M/Q_j) = \{\mathfrak{p}_j\}$ for all $j \in \{1, \ldots, m\}$. Furthermore, the set of primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is independent of the choice of minimal primary decomposition and is exactly equal to $\operatorname{Ass}(M/N)$.*

In the main text of this thesis, we frequently apply this decomposition to principal ideals in unique factorization domains.

Corollary C.19. *Suppose that R is additionally a unique factorization domain and $f = p_1^{e_1} \cdots p_k^{e_k}$ is a prime factorization of a non-zero, non-unit element $f \in R$ into pairwise non-associate prime elements p_i. Then the minimal primary decomposition of the principal ideal (f) is given by*

$$(f) = (p_1^{e_1}) \cap \cdots \cap (p_k^{e_k}).$$

In particular, $\operatorname{Ass}(R/(f)) = \{(p_1), \ldots, (p_k)\}$.

Appendix D

Gelfand Transform

The goal of this appendix is to provide some background on the Gelfand transform for commutative Banach algebras. We intentionally keep the discussion focused and brief, limiting our scope to what is strictly necessary for this thesis. For proofs of the results stated here, as well as a more elaborate treatment of the theory, we refer the reader to [DE14], [Fol16], and [Arv02].

Definition D.1. A *Banach algebra* is an algebra $\mathcal{A}$ over the field of complex numbers $\mathbb{C}$ equipped with a norm $\|\cdot\|$ that makes it a complete normed vector space (a Banach space) and additionally satisfies the sub-multiplicative property:

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in \mathcal{A}$. We call $\mathcal{A}$ *unital* if it contains a multiplicative identity, and *commutative* if its multiplication is commutative.

We note that in any unital Banach algebra $\mathcal{A}$ with multiplicative identity $1_{\mathcal{A}}$ and norm $\|\cdot\|$, one may find an equivalent norm $\|\cdot\|'$ such that $(\mathcal{A}, \|\cdot\|')$ is again a Banach algebra and $\|1_{\mathcal{A}}\|' = 1$. We thus assume without loss of generality that $\|1_{\mathcal{A}}\| = 1$ in any unital Banach algebra $\mathcal{A}$.

We provide three fundamental examples of Banach algebras.

Example D.2 (Group Algebras)**.** Let G be a locally compact group and λ_G a left Haar measure on G. The space $L^1(G, \lambda_G)$ of complex-valued integrable functions on G is a Banach algebra when equipped with the convolution product

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\lambda_G(y),$$

for $f, g \in L^1(G, \lambda_G)$, and the norm $\|f\|_1 = \int_G |f(x)|\, d\lambda_G(x)$ for $f \in L^1(G, \lambda_G)$. The Banach algebra $L^1(G, \lambda_G)$ is unital if and only if G is discrete, in which case the unit is given by the characteristic function $\mathbb{1}_{\{1_G\}}$ of the singleton set $\{1_G\}$. The Banach algebra $L^1(G, \lambda_G)$ is commutative if and only if G is abelian.

Example D.3 (Function Algebras)**.** Let X be a compact Hausdorff space. The space $C(X)$ of continuous complex-valued functions on X is a Banach algebra when equipped with the pointwise product and the supremum norm $\|f\|_\infty = \sup_{x \in X}|f(x)|$ for $f \in C(X)$. The Banach algebra $C(X)$ is unital, with unit given by the constant function $\mathbb{1}_X$. The Banach algebra $C(X)$ is commutative.

Example D.4 (Operator Algebras)**.** Let E be a non-trivial Banach space. The space $\mathcal{B}(E)$ of bounded linear operators on E is a Banach algebra when equipped with operator composition as the multiplication operation and the operator norm $\|T\| = \sup_{\|x\|\leq 1}\|T(x)\|$ for $T \in \mathcal{B}(E)$. The Banach algebra $\mathcal{B}(E)$ is unital, with unit given by the identity operator id_E. The Banach algebra $\mathcal{B}(E)$ is commutative if and only if E is one-dimensional.

For the remainder of this appendix, we fix a commutative, unital Banach algebra $\mathcal{A}$.

Definition D.5. A *character* of $\mathcal{A}$ is a non-zero algebra homomorphism $\omega : \mathcal{A} \to \mathbb{C}$. The set of all characters of $\mathcal{A}$ is called the *Gelfand spectrum* of $\mathcal{A}$ and is denoted by $\Delta_{\mathcal{A}}$.

Proposition D.6. *For any $\omega \in \Delta_{\mathcal{A}}$, we have $\|\omega\| = 1$, where $\|\omega\|$ denotes the operator norm of $\omega : \mathcal{A} \to \mathbb{C}$. In particular, ω is bounded, and hence continuous.*

The above proposition establishes that $\Delta_{\mathcal{A}}$ is a subset of the continuous dual $\mathcal{A}^*$ of bounded linear functionals on $\mathcal{A}$. We can thus equip $\Delta_{\mathcal{A}}$ with the subspace topology inherited from the weak*-topology on $\mathcal{A}^*$, which is the coarsest topology on $\mathcal{A}^*$ that makes all evaluation maps $f \mapsto f(a)$ from $\mathcal{A}^*$ to $\mathbb{C}$, for all $a \in \mathcal{A}$, continuous.

Proposition D.7. *Equipped with the topology described above, $\Delta_{\mathcal{A}}$ is a compact Hausdorff space.*

Definition D.8. For any $a \in \mathcal{A}$, we define the function $\hat{a}$ on the Gelfand spectrum by

$$\begin{aligned} \hat{a} : \Delta_{\mathcal{A}} &\to \mathbb{C} \\ \omega &\mapsto \omega(a). \end{aligned}$$

The map $a \mapsto \hat{a}$ is called the *Gelfand transform.*

Theorem D.9. *The Gelfand transform is a norm-non-increasing algebra homomorphism from $\mathcal{A}$ to $C(\Delta_{\mathcal{A}})$, the Banach algebra of continuous complex-valued functions on $\Delta_{\mathcal{A}}$ equipped with the supremum norm.*

List of Symbols

$\mathbb{1}_A$	The characteristic (indicator) function of a set A (Proposition 1.18).
$\langle x, \chi \rangle$	The canonical duality pairing $X \times \widehat{X} \to \mathbb{S}^1$ evaluating a character χ at a point x (Theorem A.6).
$\mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}$	Fraktur letters are used to denote ideals in a ring, with $\mathfrak{p}$ and $\mathfrak{q}$ typically reserved for prime ideals (Section 3.2 and Appendix C).
α	An action of a group, usually a quasi-algebraic or algebraic action (Definitions 1.1 and 3.1).
α_γ	The automorphism associated with a group element $\gamma \in \Gamma$ (Definition 1.1).
$\alpha_\Gamma(x)$	The orbit of a point x under an action α (Definition 1.5).
$\alpha^Y, \alpha^{X/Y}$	The restricted action on a closed α-invariant subgroup $Y \subseteq X$ and the induced action on the quotient X/Y.
α^M	The algebraic $\mathbb{Z}^d$-action dual to an R_d-module M (Proposition 3.2).
$\mathrm{Ann}_R(M)$	The annihilator ideal of a module M over a ring R (Appendix C).
$\mathrm{Ass}_R(M)$	The set of associated prime ideals of a module M over a ring R (Appendix C).
$\mathrm{Aut}(X)$	The group of bi-continuous automorphisms of a topological group X (Definition 1.1).
$A \triangle B$	The symmetric difference of sets A and B (Lemma 1.17).
$B^\perp$	The annihilator of a closed subgroup B of a locally compact abelian group A (Proposition A.7).
$\mathcal{B}_X$	The Borel σ-algebra of a topological space X (Section 1.2).
$[c]_F$	A cylinder set defined by a finite window $F \subseteq \Gamma$ and a configuration $c \in G^F$ on a product space G^Γ (Example 1.14).
$c_f(\mathbf{n})$	The integer coefficient of the monomial $u^{\mathbf{n}}$ in a Laurent polynomial $f \in R_d$ (Proposition 3.2).
$\mathrm{char}(R)$	The characteristic of a ring R (Definition 3.9).
$\Delta_{\mathcal{A}}$	The Gelfand spectrum (space of characters) of a commutative Banach algebra $\mathcal{A}$ (Appendix D).
d_G	A compatible left-invariant metric on a locally compact group G (Theorem A.2).
$\overline{\mathbb{F}}_p$	An algebraic closure of the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (with the convention that $\overline{\mathbb{F}}_0$ denotes an algebraic closure of $\mathbb{Q}$) (Definition 3.9).
Γ	A countably infinite group, often the acting group (Definition 1.1).
$h(\alpha)$	The dynamical (topological/metric) entropy of an action α (Outlook).
$\mathrm{I}(A)$	The ideal of polynomials in d variables with coefficients in a field k vanishing on a subset $A \subseteq k^d$ (Section C.2).
λ_X	The normalized Haar measure on a compact group X (Theorem A.3).
$L^2(X, \lambda_X)$	The Hilbert space of square-integrable complex-valued functions on a measure space (X, λ_X) (Proposition 1.21).

$L_0^2(X, \lambda_X)$	The subspace of $L^2(X, \lambda_X)$ consisting of functions with mean zero (Proposition 1.21).
$\mathbb{M}(f)$	The Mahler measure of a Laurent polynomial $f \in R_d$ (Outlook).
π_S	The canonical projection from a product space G^Γ onto the coordinates of a subset $S \subseteq \Gamma$ (Definition 2.10).
$\hat{\varphi}$	The continuous group homomorphism dual to a continuous group homomorphism φ (Proposition A.11).
$\operatorname{rad}(\mathfrak{a})$	The radical of an ideal $\mathfrak{a}$ (Section C.2).
R_d	The ring of Laurent polynomials $\mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ (Section 3.1).
σ	The shift-action on a product space G^Γ (Definition 2.1).
$\mathbb{S}^d$	The product $(\mathbb{S}^1)^d \subseteq \mathbb{C}^d$ of d copies of the unit circle $\mathbb{S}^1$ in $\mathbb{C}$ (Theorem 3.17).
T_A	Arnold's Cat Map on the 2-torus, induced by the matrix A (Example 0.1).
T_R	The Toral Rotation map on the 2-torus (Example 0.2).
T_2	The Times Two Map on the 1-torus (Example 0.3).
$\mathbb{T}^d$	The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ (Example 0.1).
$u^{\mathbf{n}}$	The monomial $u_1^{n_1} \dots u_d^{n_d}$ in R_d for $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ (Section 3.1).
$\mathcal{U}(\mathcal{H})$	The group of unitary operators on a complex Hilbert space $\mathcal{H}$ (Appendix B).
$\mathrm{V}(\mathfrak{p})$	The algebraic variety associated to a prime ideal $\mathfrak{p} \subseteq R_d$ (Definition 3.9).
$\mathrm{V}_{\mathbb{C}}(\mathfrak{a})$	The complex variety associated to an ideal $\mathfrak{a} \subseteq R_d$ (Definition 3.9).
X, Y	Compact metrizable groups (required to be abelian in the context of algebraic $\mathbb{Z}^d$-actions) (Definitions 1.1 and 3.1).
$\widehat{X}$	The Pontryagin dual of a locally compact abelian group X, or the unitary dual of a locally compact group X (Theorem A.6 and Appendix B).
X^M	The compact metrizable abelian group dual to a countable R_d-module M (Proposition 3.2).

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