Locally uniformly convergent sequence of holomorphic functions:
Let $(f_n)_{n\geq 1} \subseteq \mathcal{H}(\Omega)$, Ω open, that converge uniformly on compacta to $f: \Omega \to \mathbb{C}$.

Then f is holomorphic and $(f_n')_{n\geq 1}$ converge uniformlyon compacta to f'.

holomorphic_locally_uniformly_convergent_results_in_holomorphic derivative_of_locally_uniformly_convergent

Weierstrass M-Test: Weierstrass_M-test
Let $f_n : \Omega \to \mathbb{C}$ holomorphic, Ω open and $U \subseteq \Omega$ open. Suppose there exists a sequence of real numbers $M_n \geq 0$ such that $\forall z \in U$

$$|f_n(z)| \leq M_n \quad \text{and} \quad \sum_{n\geq 1} M_n < \infty$$

Then $\sum_{n\geq 1} f_n$ converges uniformly on U.

Let $f \in C^0(\Omega, \mathbb{C})$, where $\Omega \subseteq \mathbb{C}$ is open:

$$f \in \mathcal{H}(\Omega) \iff \forall \text{ Triangle } T \subseteq \Omega : \int_T f\, dz = 0$$

Complex Integration:
The Integral of a complex function is defined as the integral along a parameterized_curve as in complex_path_integral.

Goursat: Goursat_Theorem
If $f: \Omega \to \mathbb{C}$ is holomorphic then for every Triangle $T \subseteq \Omega$

$$\int_T f\, dz = 0$$

Goursat_Theorem_prime tells us that this even holds if $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, f is continuous on Ω and any rectangle $R \subseteq \Omega$

Morera: Moreras_theorem
Let $\Omega \subseteq \mathbb{C}$ open, $f: \Omega \to \mathbb{C}$ continuous such that for any open disc $D \subseteq \Omega$ and for any triangle whose inside is also in D we have

$$\int_T f\, dz = 0$$

Then f is holomorphic on Ω.

Properties of Zeros: properties_order_holomorphic_function

Let Ω be open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$. Then
(i) If $\mathrm{ord}_{z_0}(f) = \infty$ then $\exists r > 0 : f(z) = 0$ for any $z \in D_r(z_0)$ such that $D_r(z_0) \subset \Omega$ (ie f is locally zero).
(ii) If $\mathrm{ord}_{z_0}(f) \neq \infty$ then $\exists! h \in \mathcal{H}(D_r(z_0))$ and $n \in \mathbb{Z}, n \geq 0$ such that

$$f(z) = (z - z_0)^n h(z), \qquad \forall z \in D_r(z_0)$$

where $h(z_0) \neq 0$ and $n = \mathrm{ord}_{z_0}(f)$.
(iii) For any $g \in \mathcal{H}(\Omega)$ we have

$$\mathrm{ord}_{z_0}(f+g) \geq \min(\mathrm{ord}_{z_0} f, \mathrm{ord}_{z_0} g)$$
$$\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$$

Parameteringegral: parameterintegral_for_holomorphic_functions
Let $\Omega \subseteq \mathbb{C}$ be open, $I = [a, b] \subseteq \mathbb{R}$ closed and bounded. Let $F: \Omega \times I \to \mathbb{C}$ be a function such that

- F is continuous on $\Omega \times I$.
- For each $t_0 \in I$ the function $f_{t_0}(z) := F(z, t_0) : \Omega \to \mathbb{C}$ is holomorphic.

Then the function $f(z)$ defined by the integral

$$f(z) := \int_a^b F(z, t)\, dt$$

is holomorphic and $f'(z) = \int_a^b F'(z, t)\, dt$.

If $f: \Omega \to \mathbb{C}$ is a continuous function that has a primitive_of_a_complex_function then we have independence_of_path_between_two_points

Cauchy's Estimate:
Cauchys_estimate_for_derivatives_of_holomorphic_functions
Let $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$, $r > 0$, $\overline{D_r(z_0)} \subseteq \Omega$, then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}$$

Holomorphic Function: holomorphic_function
Let $f: \Omega \to \mathbb{C}$ a function, $\Omega \subseteq \mathbb{C}$ open, $z_0 \in \Omega$. We say f is *holomorphic at* z_0 if the following limit exists

$$\lim_{z \to z_0, z \neq z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0, h \neq 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Integral over closed loop is zero integral_of_closed_loop_of_func tion_with_primitive

Cauchy Lemma:
existence_of_primitive_of_continuous_function_with_zero_integral_on_rectangles
Let D be an open disc and let f continuous in D with the property that

$$\int_R f\, dz = 0$$

for every rectangle R, whose sides are parallel to the axes. Then f has a primitive.

Liouville's Theorem: Liouvilles_theorem
If $f \in \mathcal{H}(\mathbb{C})$ and bounded, then f is constant.

If $f \in \mathcal{H}(\mathbb{C})$ non-constant, then $\mathrm{Im}(f)$ is dense in $\mathbb{C}$. Funktionentheorie_Serie06

Power Series Expansion:
local_power_series_expansion_of_holomorphic_function
Let f be holomorphic on an open set Ω, $z_0 \in \Omega$ and $r > 0$, so that $\overline{D_r(z_0)} \subseteq \Omega$. Then f has a power series expansion, i.e. $\forall z \in D_r(z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{with} \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

power_series define a holomorphic function in their radius of convergence absolute_convergence_of_power_series as in power_series_define_holomorphic_function

If $f = u + iv : \Omega \to \mathbb{C}$, Ω open and $u, v \in C^1$, then f is holomorphic iff the Cauchy-Riemann_equations hold

Cauchy: Cauchys_Theorem
If $f \in \mathcal{H}(\Omega)$ and $\gamma \subseteq \Omega$ closed curve, then

$$\int_\gamma f\, dz = 0$$

This even holds if there is a point $z_0 \in \Omega$ where f is merely continoous Cauchys_Theorem_prime

Cauchy's Integral Formula: Cauchy_integral_formula, Cauchy_integral_formula_for_dervatives
Let $f \in \mathcal{H}(\Omega)$ and $D \subseteq \Omega$ a closed disc, set $C = \partial D$, then $\forall z \in D$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z)^{n+1}}\, dw$$

Fundamental Theorem of Algebra:
Fundamentalsatz_der_Algebra
A non-constant polynomial of degree $n \geq 1$ has exactly n roots.

Isolated Zeroes: isolated_zero
For $f \in \mathcal{H}(\Omega)$ (Ω open) with $0 < \mathrm{ord}_{z_0} f < \infty$ we have a $\delta > 0$ such that $\forall z \in D_\delta^*(z_0)$

$$f(z) \neq 0$$

i.e. the zero at z_0 is isolated.

Identity Theorem: identity_theorem
Let $f, g \in \mathcal{H}(\Omega)$, where $\Omega \neq \varnothing$ is open and connected. Then the following are equivalent:
(i) $f \equiv g$ on Ω
(ii) $\exists a \in \Omega$ such that $f^{(n)}(a) = g^{(n)}(a)\, \forall n \geq 0$
(iii) $\{z \in \Omega \mid f(z) = g(z)\}$ has a limit point in Ω

Holomorphic Functions on a domain are an integral domain:
zero_divisors_of_holomorphic_functions_on_connected_do main, Integritaetsring

Image around essential singularity:
Let $f \in \mathcal{H}(D_r^*(z_0))$ and let f have an essential singularity at z_0. Then Caseroti-Weierstrass asserts that the image of $D_r^*(z_0)$ is dense in $\mathbb{C}$. And Great-Picard makes an even stronger statement that $\mathbb{C} \setminus f(D_r^*(z_0))$ contains at most one point.

Characterisation of Poles: characterisations_of_a_pole_of_a_complex_function
For $m \in \mathbb{N}$, $m \geq 1$ the following statements about $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ are equivalent.
(i) f has a pole of order m at z_0
(ii) $\exists r > 0$, $g \in \mathcal{H}(D_r(z_0))$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in D_r^*(z_0)$
(iii) $\exists r > 0$ such that $D_r(z_0) \subseteq \Omega$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z) \neq 0$ for all $z \in D_r^*(z_0)$, h has a zero of order m at z_0 and such that

$$f(z) = \frac{1}{h(z)}$$

Riemann Continuation Theorem: Riemann_continuation_theorem
Let $z_0 \in \mathbb{C}$ and $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, Ω open. Then the following are equivalent.
(i) f is holomorphically extendable to all of Ω.
(ii) f is continually extendable to all of Ω.
(iii) f is bounded in a neighbourhood of z_0, i.e. $\exists r > 0$ such that f is bounded in $D_r^*(z_0)$.
(iv) $\lim_{z \to z_0} (z - z_0) f(z) = 0$

Homotopy Theorem: Homotopy_Theorem
Let $\Omega \subseteq \mathbb{C}$ open, $f \in \mathcal{H}(\Omega)$, γ_0, γ_1 two curves in Ω that are either
(i) γ_0, γ_1 closed curves $\gamma_0 \sim_\Omega \gamma_1$.
(ii) γ_0, γ_1 are homotopic with fixed end points.
Then

$$\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz$$

Residue Theorem: Residue_theorem, General_Residue_Theorem
Let $\Omega \subseteq \mathbb{C}$ be open, $F = \{z_1, \ldots, z_n\}$ a finite set in Ω. Suppose $f \in \mathcal{H}(\Omega \setminus F)$. Let γ be any circle contained in Ω, with positive (counter clockwise) orientation such that $F \cap \gamma = \varnothing$. Let D be the open disc bounded by γ. Then

$$\int_\gamma f\, dz = 2\pi i \sum_{z_i \in F \cap D} \mathrm{res}_{z_i} f$$

This can be generalized to $\Omega \subseteq \mathbb{C}$ be simply connected, $f \in \mathcal{M}(\Omega)$, $V = \Omega \setminus S_f$. Let γ be a closed curve in V. Then we have

$$\int_\gamma f\, dz = 2\pi i \sum_{z_0 \in S_f} w_\gamma(z_0) \cdot \mathrm{res}_{z_0}(f)$$

Logarithmic Derivative:
logarithmic_derivative_of_meromorphic_function
Let $\Omega \subseteq \mathbb{C}$ be open and connected, $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$. Then f'/f, the "logarithmic derivative" of f, is also in $\mathcal{M}(\Omega)$. And f'/f has poles of order 1 at points $z_0 \in \Omega$ for which $\mathrm{ord}_{z_0} f \neq 0$, i.e. f'/f has poles at zeroes and poles of f. Furthermore

$$\mathrm{res}_{z_0} \frac{f'}{f} = \mathrm{ord}_{z_0} f$$

Power Series Expansion around Pole:
local_power_series_expansion_with_poles
If $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ has a pole of order n at z_0. Then $\exists r > 0$ such that

$$f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}}_{\text{Principle part of } f \text{ at the pole } z_0} + G(z)$$

such that $G(z) \in \mathcal{H}(D_r(z_0))$.
We call a_{-1} the *residue* of f at the pole $z = z_0$. $\mathrm{res}_{z_0} f = a_{-1}$. We also denote the pirnciple part of f at z_0 with $P_{z_0}(f)$.

Calculating the residue: Lemma_for_calculating_residues
If $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ has a pole of order n at z_0.Then

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} ((z - z_0)^n f(z))$$

Holomorphic on simply connected domain has primitive:
holomorphic_function_on_simply_connected_domain_has_primitive
Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $f \in \mathcal{H}(\Omega)$. Then f has a primitive and in particular

$$\int_\gamma f\, dz = 0$$

for any closed curve γ.

Logarithm of Function: logarithm_of_complex_function
Let $f \in \mathcal{H}(\Omega)$ non-vanishing, $\Omega \subseteq \mathbb{C} \setminus \{0\}$ simply connected. Then there exists a holomorphic function $g: \Omega \to \mathbb{C}$ such that

$$f(z) = \exp(g(z))$$

Argument Principle: argument_principle
Let $\Omega \subseteq \mathbb{C}$ be open and connected, $\partial D = \gamma \subseteq \Omega$ a circle (or any curve for which the Residue theorem applies) such that $D \subseteq \Omega$. Let $f \in \mathcal{M}(\Omega)$ with no zeroes or poles on γ. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = \sum_{z_0 \in Z_f \cap D} \mathrm{ord}_{z_0} f + \sum_{z_0 \in S_f \cap D} \mathrm{ord}_{z_0} f$$

where Z_f is the set of zeroes and S_f the set of poles.

Rouché's Theorem: Rouches_Theorem

Let $f, g \in \mathcal{H}(\Omega)$, $\Omega \subseteq \mathbb{C}$ open, containing a cricle C and its interior and $\forall z \in C$

$$|g(z)| < |f(z)|$$

Then f and $g + f$ has the same number of zeroes inside C (accounting for multiplicity).

Open Mapping Theorem:
Open_mapping_theorem_for_holomorphic_functions
Let Ω be open and connected $f \in \mathcal{H}(\Omega)$, f is not constant. Then f is an open map (i.e. sends open sets to

Schwarz's Lemma: Schwarzs_Lemma
Let $f: \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then:
(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
(ii) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$ then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$, i.e. f is a rotation.
(iii) $|f'(0)| \leq 1$ and equality holds if any only if $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Existence of Logarithm:
branch_of_complex_logarithm_on_simply_connected_domains
Let $\Omega \subseteq \mathbb{C} \setminus \{0\}$ be a simply connected set. Then there exists a branch of logarithm on Ω, i.e. a holomorphic function $F \in \mathcal{H}(\Omega)$ such that $\forall z \in \Omega : \exp(F(z)) = z$.

Riemann Mapping Theorem: Riemann_Mapping_Theorem
If $\Omega \subseteq \mathbb{C}$, $\Omega \neq \varnothing$, $\Omega \neq \mathbb{C}$, Ω simply connected and $z_0 \in \Omega$. Then there exists a unique $F: \Omega \to \mathbb{D}$ conformal equivalence such that $F(z_0) = 0$, $F'(z_0) > 0$ (i.e. $F'(z_0)$ is real).

Maximum Modulus Principle: maximum_modulus_principle
Let Ω be open and connected, $f \in \mathcal{H}(\Omega)$, non-constant. Then there is no $z_0 \in \Omega$ such that $\forall z \in \Omega$

$$|f(z)| \leq f(z_0)$$

i.e. f cannot attain its maximum inside Ω.
In particular if $\overline{\Omega}$ is bounded and f is continuous on $\overline{\Omega}$, then

$$\max_{z \in \overline{\Omega}} |f(z)| \leq \max_{z \in \overline{\Omega} \setminus \Omega} |f(z)|$$

the LHS exists since f is continuous on $\overline{\Omega}$, $\overline{\Omega}$ is compact.

Automorphisms of the unit disc: automorphisms_of_unit_disc
Let $f: \mathbb{D} \to \mathbb{D}$ be an automorphism of $\mathbb{D}$. Then $\exists \theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

then $f(0) = e^{i\theta}\alpha$, $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$. Conversly any map of the form $e^{i\theta}\left(\frac{\alpha - z}{1 - \bar{\alpha} z}\right)$ is an automorphism of $\mathbb{D}$.